MULTIPLE INTERPHASES FOR FRACTIONAL ALLEN–CAHN EQUATIONS

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Abstract. We consider a nonlocal reaction-diffusion equation in $\mathbb{R}^n$, $n \geq 2$, that physically describes dislocations in crystalline structures. In particular, we study the evolutionary version of the classical Peierls–Nabarro model with initial conditions corresponding to multiple slip loop dislocations. After suitably rescaling the equation with a small phase parameter $\varepsilon > 0$, the rescaled solution solves a fractional Allen–Cahn equation. We show that, as $\varepsilon \to 0$, the limiting solution exhibits multiple interfaces evolving independently and according to their mean curvature.

1. Introduction

In this paper, we study a nonlocal, reaction-diffusion equation that arrises naturally in the Peierls–Nabarro model for atomic dislocations in crystalline structures. Our initial configuration corresponds to a collection of *slip loop dislocations* with the same orientation. After suitably rescaling the problem from the microscopic scale to the mesoscopic scale, we show that the dislocation loops move independently, according to their mean curvature.

At the atomic level, one can view a crystal as an infinite cubic lattice. Dislocations are defects from a perfect lattice which evolve when subject to forces, see [17]. The evolution of *edge dislocations* has been recently studied in the literature, that is, when the dislocations are straight, parallel lines. In this special setting, the Peierls–Nabarro model reduces to a *one-dimensional* PDE and the dislocation lines can be associated to single points in $\mathbb{R}$. At the mesoscopic scale, González–Monneau in [16] showed that the dislocation points evolve according to a discrete system of ODEs. See [12] for an overview of the latest theory. Nevertheless, when the dislocations are not straight edge dislocations, the physical model can only be reduced to a *two-dimensional* PDE and the dislocations are indeed curves in $\mathbb{R}^2$. Unlike the one-dimensional setting in which dislocation points move left or right, curves in higher dimensions can move in infinitely many directions. To the best of our knowledge, we are the first to study the movement of dislocation curves in $\mathbb{R}^n$, $n \geq 2$.

Before further reviewing the Peierls–Nabarro model for slip loop dislocations, let us formalize our problem mathematically.

1.1. Setting of the problem.

We are interested in the fractional Allen–Cahn equation

$$\varepsilon \partial_t u^\varepsilon = \frac{1}{\varepsilon\,|\ln \varepsilon|}\left(\varepsilon \mathcal{I}_n[u^\varepsilon] - W'(u^\varepsilon)\right) \quad \text{in } \mathbb{R}^n \times (0,\infty),\ n \geq 2, \tag{1.1}$$

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where $\varepsilon > 0$ is a small parameter, $\mathcal{I}_n = -c_n(-\Delta)^{\frac{1}{2}}$ denotes, up to a constant, the square root of the Laplacian in $\mathbb{R}^n$, and W is a multi-well potential. The operator $\mathcal{I}_n$ is a nonlocal integro-differential operator of order 1 and is given by

$$\mathcal{I}_n u(x) = \mathrm{P.V.} \int_{\mathbb{R}^n} (u(x+y) - u(x)) \frac{dy}{|y|^{n+1}} dy, \quad x \in \mathbb{R}^n, \quad n \in \mathbb{N}, \tag{1.2}$$

where P. V. indicates that the integral is taken in the principal value sense. For further background on fractional Laplacians, see for example [10, 30]. Regarding the potential W, we assume that

$$\begin{cases} W \in C^{4,\beta}(\mathbb{R}) & \text{for some } 0 < \beta < 1 \\ W(u+1) = W(u) & \text{for any } u \in \mathbb{R} \\ W = 0 & \text{on } \mathbb{Z} \\ W > 0 & \text{on } \mathbb{R} \setminus \mathbb{Z} \\ W''(0) > 0. \end{cases} \tag{1.3}$$

We let u^ε be the solution to (1.1) when the initial condition u_0^ε is a superposition of layer solutions. The layer solution (also called the phase transition) $\phi : \mathbb{R} \to [0,1]$ is the unique solution to

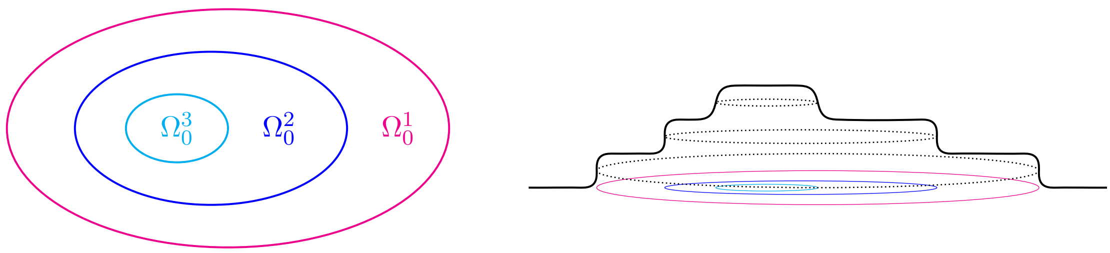
$$\begin{cases} C_n \mathcal{I}_1[\phi] = W'(\phi) & \text{in } \mathbb{R} \\ \dot{\phi} > 0 & \text{in } \mathbb{R} \\ \phi(-\infty) = 0, \quad \phi(+\infty) = 1, \quad \phi(0) = \frac{1}{2}, \end{cases} \tag{1.4}$$

where $\mathcal{I}_1$ is the fractional operator in (1.2) on $\mathbb{R}$ and the constant $C_n > 0$ (given explicitly in (3.1)) depends only on $n \geq 2$. Further discussion on ϕ will be presented in Section 4.

For a fixed $N \in \mathbb{N}$, let $(\Omega_0^i)_{i=1}^N$ be a finite sequence of open subsets of $\mathbb{R}^n$ that are smooth, bounded, and satisfy $\Omega_0^{i+1} \subset\subset \Omega_0^i$. The corresponding boundaries $\Gamma_0^i = \partial \Omega_0^i$ can be understood as the initial dislocation loops in the crystal. Let $d_i(x)$ be the signed distance function associated to Γ_0^i, $i = 1, \dots, N$, given by

$$d_i(x) = \begin{cases} d(x, \Gamma_0^i) & \text{if } x \in \Omega_0^i \\ -d(x, \Gamma_0^i) & \text{otherwise.} \end{cases} \tag{1.5}$$

For our initial condition to be well-prepared, we let u_0^ε be the N-fold sum of the layer solutions $\phi(d_i(x)/\varepsilon)$, see Figure 1.



(A) Initial sets in $\mathbb{R}^n$ (B) Initial condition u_0^ε for small $\varepsilon > 0$

FIGURE 1. Initial configuration for $N = 3$

We show that the evolution of the fronts, denoted by $(\Gamma_t^i)_{t \geq 0}$, corresponds to interfaces moving independently by mean curvature meaning that the sets $(\Gamma_t^i)_{t \geq 0}$ move with normal velocity

$$v = \mu \sum_{i=1}^{n-1} \kappa_i, \quad \mu := \frac{c_0}{2} \frac{|S^{n-2}|}{n-1} > 0, \tag{1.6}$$

where κ_i are the principal curvatures and $c_0 > 0$ is explicit (see (4.1)). To handle possible singularities, we use the level set approach: Γ_t^i is the zero level set at time $t > 0$ of a solution u^i to the mean curvature equation whose zero level set at $t = 0$ is exactly Γ_0^i. In this case, we say that $({}^+\Omega_t^i, \Gamma_t^i, {}^-\Omega_t^i)$ denotes the level-set evolution of $(\Omega_0^i, \Gamma_0^i, (\overline{\Omega_0^i})^c)$ where ${}^+\Omega_t^i$ and ${}^-\Omega_t^i$ are the positivity and negativity sets of u^i respectively. See Section 2 for more definitions, details, and references on the level set approach to motion by mean curvature.

We now present the main result of our paper.

Theorem 1.1. *Let $u^\varepsilon = u^\varepsilon(t,x)$ be the unique solution of the reaction-diffusion equation (1.1) with the initial datum $u_0^\varepsilon : \mathbb{R}^n \to [0,N]$ defined by*

$$u_0^\varepsilon(x) = \sum_{i=1}^{N} \phi\left(\frac{d_i(x)}{\varepsilon}\right). \tag{1.7}$$

Then, as $\varepsilon \to 0$, the solutions u^ε satisfy

$$\begin{cases} u^\varepsilon \to N & \text{in } {}^+\Omega_t^N, \\ u^\varepsilon \to i & \text{in } {}^+\Omega_t^i \cap {}^-\Omega_t^{i+1}, \quad i = 1, \dots, N-1, \\ u^\varepsilon \to 0 & \text{in } {}^-\Omega_t^1, \end{cases}$$

where $({}^+\Omega_t^i, \Gamma_t^i, {}^-\Omega_t^i)$ denotes the level-set evolution of $(\Omega_0^i, \Gamma_0^i, (\overline{\Omega_0^i})^c)$ in which $(\Gamma_t^i)_{t\geq 0}$ locally move with normal velocity (1.6).

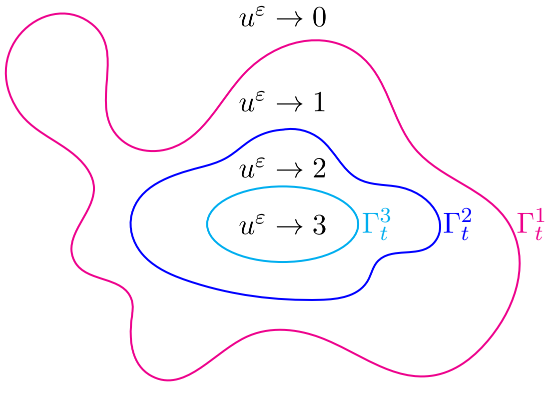


Figure 2. Convergence result for $N = 3$

As illustrated in Figure 2, Theorem 1.1 says that the solutions u^ε converge to integers between the interfaces Γ_t^i but does not say anything about the limiting solution on the curves themselves. To understand this, we say that the set Γ_t^i does not develop interior if and only if $\Gamma_t^i = \partial({}^+\Omega_t^i) = \partial({}^-\Omega_t^i)$. In this special setting, the limiting function in Theorem 1.1 makes integer jumps on the curves Γ_t^i and satisfies

$$\lim_{\varepsilon\to 0} u^\varepsilon = \frac{N}{2} + \frac{1}{2}\sum_{i=1}^{N}\left(\mathbb{1}_{{}^+\Omega_t^i} - \mathbb{1}_{(\overline{{}^+\Omega_t^i})^c}\right) \quad \text{in } (0,\infty)\times\mathbb{R}^n \setminus \bigcup_{i=1}^{N}\Gamma_i$$

where $\mathbb{1}_\Omega$ denotes the characteristic function of the set $\Omega \subset \mathbb{R}^n$. However, due to the degeneracy of the mean curvature equation, Γ_t^i may develop interior, and we cannot say exactly where the jump occurs within these sets. For example, the level set evolution of a dumbbell will develop singularities on its neck in finite time.

Theorem 1.1 for $N = 1$ has been addressed in the literature when W is instead a double-well potential. The classical Allen–Cahn equation for which (1.1) is instead driven by the usual Laplacian Δ was studied famously by Modica–Mortola [20] for the stationary case.

Chen studied the corresponding evolutionary Allen–Cahn problem and proved that the solution exhbits an interface moving by mean curvature [6]. In the fractional setting, the stationary case was studied by Savin–Valdinoci in [29] and the evolution problem was considered by Imbert–Souganidis in the preprint [19]. See Section 2 for more on the phase field theory. We are the first to study when $N > 1$ and W is a multi-well potential.

The proof of Theorem 1.1 relies on the abstract method introduced in [3] for the study of front propagation. One of the key tools needed for the abstract method is the construction of strict sub/super solutions to (1.1). Since we are working with multiple evolving fronts, our barriers roughly take the form

$$v^\varepsilon(t,x) = \sum_{i=1}^{N} \phi\left(\frac{d_i(t,x)}{\varepsilon}\right) + \text{lower order correctors}$$

where $d_i(t,\cdot)$ is the signed distance function associated to Γ_t^i and ϕ is the solution to (1.4). A formal argument for this choice of barrier is presented in Section 5.

One difficulty arises in understanding $v^\varepsilon(t,x)$ when (t,x) is far from the front Γ_t^i since the signed distance function is not smooth at such points. To overcome this, we replace d_i with a smooth extension of the signed distance function away from the curve, see Definition 4.3. In [19], they instead extend the barrier $\phi(d_1/\varepsilon)$ (for $N = 1$) away from the front. We found that a superposition of the corresponding barriers in [19] cannot be applied in this setting as we cannot control the errors associated with the nonlinearity of the potential. By extending the signed distance functions d_i instead of each $\phi(d_i/\varepsilon)$, we are able to use the asymptotic behavior of ϕ at $\pm\infty$ to show that v^ε is indeed a sub/super solution (see Section 6).

In the actual construction of the barrier, we add lower order correctors to control the error as $\varepsilon \to 0$. For this, we use a superposition of solutions $\psi = \psi_\varepsilon$ to the linearized equation

$$-C_n\mathcal{I}_1[\psi] + W''(\phi)\psi = g$$

for some right-hand side $g = g_\varepsilon$ depending $\varepsilon > 0$ and on the signed distance function to the corresponding front. The explicit form of g is somewhat technical and is given in Section 4 with a heuristical derivation in Section 5. In [19], they assume the existence and uniqueness of ψ and also that ψ and its derivatives are bounded independent of ε. We complete the story by proving existence and uniqueness, but have found that ψ cannot be bounded independently of $\varepsilon > 0$. Indeed, we can only show, via delicate analysis, that the corrector $\psi = \psi_\varepsilon$ satisfies

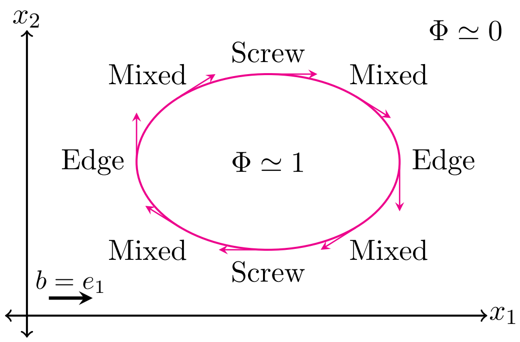
$$|\psi_\varepsilon| \leq \frac{C}{\varepsilon^{\frac{1}{2}}|\ln \varepsilon|}$$

which is enough for our arguments, see Theorem 4.9.

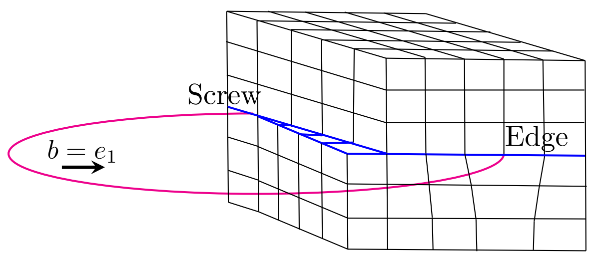
1.2. **The Peierls–Nabarro model for slip loop dislocations.** The Peierls–Nabarro model is a phase field model for dislocation dynamics which incorporates the atomic features of a crystalline structure into continuum framework [24, 25, 28]. In the phase field approach, dislocations are interfaces represented by a transition of a continuous field. We briefly review the model for slip loop dislocations (see [17, 18] for more details).

Recall that a perfect crystal can be understood as a simple cubic lattice and a dislocation is a defect from perfect atomic alignment. A slip dislocation occurs when a portion of atoms slides over another along a particular plane, called the slip plane, and is the curve created by the boundary between the shifted and unshifted regions. In Cartesian coordinates $x_1x_2x_3$, we assume that the slip plane is the x_1x_2-plane. The movement of the dislocation is determined by the so-called Burgers' vector b. For slip dislocations, the Burgers' vector is contained within the slip plane, so we assume it to be in the direction of the x_1-axis, say $b = e_1$.

There are two types of slip dislocations that correspond to straight lines in $\mathbb{R}^2$: edge and screw. An edge dislocation is formed when an extra half-plane of atoms is included in the crystal. In this case, the Burgers' vector is perpendicular to the dislocation line and the motion of the dislocation line is in the direction of b. A screw dislocation instead creates a spiral or helical path around the core. Here, the Burgers' vector is parallel to the dislocation curve which means that the dislocation line moves parallel to the direction of b. It is probable that most slip dislocations are not straight lines and thus exhibit components of both edge and screw dislocations. We call these mixed dislocations. *Slip loops* (or loop dislocations) are slip dislocations in the form of simple closed curves. Since the Burgers' vector is fixed for a given slip loop, the type of dislocation changes from point to point along the dislocation curve, see Figure 3. Depending on the orientation of the curve with respect to b, the slip loop will either shrink or expand.



(A) In the slip plane x_1x_2



(B) In $\mathbb{R}^3$ (inspired by Fig. 3.11 in [21])

FIGURE 3. Dislocation types in a slip loop dislocation with fixed Burgers' vector

In order to describe the loop, a phase parameter $\Phi(x_1, x_2)$ between 0 and 1 is used to capture the disregistry of the upper half crystal $\{x_3 > 0\}$ from the lower half crystal $\{x_3 < 0\}$. In particular, the dislocation loop is the set $\Gamma_0 := \{\Phi = 1/2\} \subset \mathbb{R}^2$, and the phase parameter satisfies $\Phi \simeq 1$ inside the loop and $\Phi \simeq 0$ outside the loop.

Next, let $U = U(x_1, x_2, x_3)$ be the distance between an atom at location (x_1, x_2, x_3) in the upper half crystal and its rest position. Let $\Phi(x_1, x_2) = U(x_1, x_2, 0)$ be the displacement in the slip plane. Most of the mismatch of atoms from their perfect lattice structure occurs within the slip plane. To quantify this, we use a multi-well potential W satisfying (1.3). Here, the periodicity of W captures the periodicity of the crystal. In the Peierls–Nabarro model, the total energy is the elastic energy for bonds between atoms plus the energy for atomic displacement:

$$\mathcal{E}(U) = \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}^+} |U(x_1, x_2, x_3)|^2 \, dx_1 \, dx_2 \, dx_3 + \int_{\mathbb{R}^2} W(\Phi(x_1, x_2)) \, dx_1 \, dx_2.$$

The equilibrium configuration is obtained by minimizing the energy $\mathcal{E}$ with respect to U under the constraint that $\Phi \simeq 1$ in the loop and $\Phi \simeq 0$ outside the loop. The corresponding Euler-Lagrange equation is

$$\mathcal{I}_2[\Phi] = W'(\Phi) \quad \text{in } \mathbb{R}^2$$

where $\mathcal{I}_2$ is the fractional Laplacian of order 1 in $\mathbb{R}^2$. It turns out that it is enough to consider $\Phi(x_1, x_2) = \phi(d(x_1, x_2))$ where ϕ solves (1.4) in $\mathbb{R}$ and d is the signed distance function associated to the loop Γ_0. (See, for example, Lemma 3.2 with $n = 2$.)

We are interested in the evolution of multiple loop dislocations in the same slip plane, with the same Burgers' vector. For this, we use a single parameter $u(t, x)$ defined for x in the slip plane $\mathbb{R}^n$, the physical dimension being $n = 2$. The dislocation dynamics are then captured by the evolutionary Peierls–Nabarro model:

$$\partial_t u = \mathcal{I}_n[u] - W'(u) \quad \text{in } (0, \infty) \times \mathbb{R}^n, \ n \geq 2,$$

where $\mathcal{I}_n$ is the fractional Laplacian of order 1 in $\mathbb{R}^n$. At the microscopic scale, we assume that the dislocations curves are at a distance of order 1 from each other. This can be represented by the initial condition

$$u(x, 0) = \sum_{i=1}^{N} \phi\left(\frac{d_i(\varepsilon x)}{\varepsilon}\right) \quad \text{for } x \in \mathbb{R}^n, \ \varepsilon > 0,$$

where ϕ solves (1.4) in $\mathbb{R}$ and d_i is the signed distance function associated to the loop Γ_0^i.

In order to understand the movement of the dislocation curves at a larger (mesoscopic) scale, we consider the rescaling

$$u^\varepsilon(t, x) = u\left(\frac{t}{\varepsilon^2 |\ln \varepsilon|}, \frac{x}{\varepsilon}\right), \quad (t, x) \in [0, \infty) \times \mathbb{R}^n, \ \varepsilon > 0. \tag{1.8}$$

Consequently, u^ε solves (1.1) with the initial condition (1.7). Here, $\varepsilon > 0$ represents the scaling between the microscopic scale ($\sim 10^{-10}\mu m$) and the mesoscopic scale ($\sim 0.1 - 10\mu m$). The presence of the factor $|\ln \varepsilon|$ is well-known in physics, see [4, 9, 17]. Roughly speaking, it arrises from an integrability condition for the kernel of the $\frac{1}{2}$-fractional Laplacian in $\mathbb{R}^n$. See also [29].

Dislocations can be described at different scales by different models:

- atomic scale (Frenkel–Kontorova model),
- microscopic scale (Peierls–Nabarro model),
- mesoscopic scale (Discrete dislocation dynamics, motion by mean curvature)
- macroscopic scale (Elasto-visco-plasticity with density of dislocation).

As described above, our result passes from the Peierls–Nabarro model to motion by mean curvature. In the case of multiple parallel straight edge dislocations, Gonzalez–Monneau passed from the one-dimensional Peierls–Nabarro model to the discrete dislocation dynamics, see [16]. We reference the reader to [12] and the references therein for more on the dynamics of edge dislocations in one dimension. In higher dimensions, the line tension effect is stronger than the interaction between the curves, and we have movement by mean curvature as seen in Theorem 1.1. See also the work of Garrioni–Müller for a variational model in dimension two [14]. Monneau and the first author in [23] studied homogenization of the Peierls–Nabarro model to describe the elasto-visco-plasticity with a density of dislocations at the macroscopic scale.

1.3. **Organization of the paper.** The rest of the paper is organized as follows. First, in Section 2, we provide the necessary background pertaining to motion by mean curvature. Section 3 contains preliminary results on fractional Laplacians. Then, in Section 4, we establish preliminary results the phase transition ϕ, the corrector ψ, and other auxiliary functions needed for the rest of the paper. We provide heuristics for the proof of Theorem 1.1 and for the choice of barrier in Section 5. The construction of barriers is presented in Section 6. Section 7 contains the proof of Theorem 1.1. We prove some auxiliary results from Section 4 in Sections 8, 9, and 10. Lastly, we present our own proof of the main convergence result from [19] in Section 11.

1.4. **Notations.** In the paper, we will denote by $C > 0$ any universal constant depending only on the dimension n and W.

We let $B(x_0, r)$ denote a ball of radius $r > 0$ centered at $x_0 \in \mathbb{R}^n$ and let S^n denote the unit sphere in $\mathbb{R}^{n+1}$.

For $\beta \in (0, 1]$ and $k \in \mathbb{N} \cup \{0\}$, we denote by $C^{k,\beta}(\mathbb{R})$ the usual class of functions with bounded $C^{k,\beta}$ norm over $\mathbb{R}$. The class $H^{\frac{1}{2}}(\mathbb{R})$ is the set of functions $g \in L^2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|g(x) - g(y)|^2}{|x - y|^3} \, dy \, dx < \infty.$$

For multi-variable functions, we let $C^{k,\beta}_\xi(\mathbb{R})$ and $H^{\frac{1}{2}}_\xi(\mathbb{R})$ denote the classes $C^{k,\beta}(\mathbb{R})$ and $H^{\frac{1}{2}}(\mathbb{R})$ in the variable $\xi \in \mathbb{R}$, respectively.

Given a function $\eta = \eta(t, x)$, we write $\eta = O(\varepsilon)$ if there is $C > 0$ such that $|\eta| \le C\varepsilon$, and we write $\eta = o_\varepsilon(1)$ if $\lim_{\varepsilon \to 0} \eta = 0$. Given a sequence of function $u^\varepsilon(t, x)$, we write

$$\liminf_{\varepsilon \to 0} {}_* u^\varepsilon(t, x) = \liminf_{\substack{(y,s) \to (t,x) \\ \varepsilon \to 0}} u^\varepsilon(y, s)$$

and

$$\limsup_{\varepsilon \to 0} {}^* u^\varepsilon(t, x) = \limsup_{\substack{(y,s) \to (t,x) \\ \varepsilon \to 0}} u^\varepsilon(y, s).$$

2. Motion by mean curvature

In this section, we describe the level set approach for the geometric motions of the fronts. For a smooth function $u = u(t, x)$, consider a level set $\Gamma_t = \{x \in \mathbb{R}^n : u(x, t) = c\}$ of u at level $c \in \mathbb{R}$ and assume that ∇u does not vanish in a neighborhood of Γ_t. Let $d(t, x)$ denote the signed distance function to Γ_t:

$$d(t, x) = \begin{cases} d(x, \Gamma_t) & \text{for } u(t, x) \ge c \\ -d(x, \Gamma_t) & \text{for } u(t, x) < c. \end{cases}$$

Note that $n(t, x) = \nabla d(t, x)$ is normal to the curve Γ_t and satisfies $|\nabla d| = 1$ in a time-space neighborhood $\mathcal{N}$ of Γ_t. Then, as theorized by Osher–Sethian [26] and justified by Evans–Spruck in [13] for viscosity solutions, the level sets $(\Gamma_t)_{t>0}$ move with normal velocity

$$\begin{aligned} v(t, x, d(t, x)) &= -\mu \operatorname{div}(n(t, x)) n(t, x) \\ &= -\mu \Delta d(t, x) \nabla d(t, x) \quad \text{in } \mathcal{N}, \end{aligned} \tag{2.1}$$

where $\mu > 0$ is in (1.6), if and only if u is a solution in $\mathcal{N}$ to the nonlinear degenerate equation

$$\partial_t u = \mu \operatorname{tr}\left((I - \widehat{\nabla u} \otimes \widehat{\nabla u}) D^2 u \right) \tag{2.2}$$

where $\hat{p} = p/|p|$ for $p \in \mathbb{R}^n$ and $\otimes$ denotes the usual tensor product. Consequently, *all* the level sets of u move according to their mean curvature (2.1) if and only if u is a solution to the *mean curvature equation* (2.2) in the whole space $(0, \infty) \times \mathbb{R}^n$. In this regard, we say that the mean curvature equation is a *geometric* equation since if u solves (2.2), then so does $\Phi(u)$ for any smooth function $\Phi : \mathbb{R} \to \mathbb{R}$. Note that the velocity in (2.1) is the same as (1.6).

For a bounded, open set $\Omega_0 \subset \mathbb{R}^n$, set $\Gamma_0 = \partial\Omega_0$ and consider the initial triplet $(\Omega_0, \Gamma_0, (\overline{\Omega_0})^c)$. Let $u_0(x)$ be such that

$$\Omega_0 = \{x : u_0(x) > 0\} \quad \text{and} \quad \Gamma_0 = \{x : u_0(x) = 0\}.$$

If u is a solution to (2.2) with initial data $u(0, x) = u_0(x)$, then, by our previous discussion, the zero level sets of u move according to their mean curvature. In particular, we set

$$^{+}\Omega_t = \{x : u(t, x) > 0\}, \quad \Gamma_t = \{x : u(t, x) = 0\}, \quad \text{and} \quad {}^{-}\Omega_t = \{x : u(t, x) < 0\}$$

and say that $(^{+}\Omega_t, \Gamma_t, {}^{-}\Omega_t)_{t \geq 0}$ denotes the *level set evolution* of $(\Omega_0, \Gamma_0, (\overline{\Omega_0})^c)$. Under certain conditions on Ω_0 (such as smooth and convex), the sets $(\Gamma_t)_{t \geq 0}$ do not develop interior, that is, $\Gamma_t = \partial(^{+}\Omega_t) = \partial(^{-}\Omega_t)$ for $t > 0$. However, this is not guaranteed for a general initial set Ω_0, so we use weak solutions to handle possible singularities, see the next section.

Consider the special case in which the curves Γ_t are smooth and do not develop interior, at least for some time. Then, the signed distance function d is smooth, satisfies $|\nabla d| = 1$ in a neighborhood of Γ_t, and is a solution to (2.2) on Γ_t. Moreover, as a consequence of the strong comparison principle, if two level sets $\{u(t, x) = c_1\}$ and $\{u(t, x) = c_2\}$ start separated, then they remain separated for some time.

Since the inaugural work [13] for mean curvature flow, the level set approach has been adapted for more general motions. Indeed, around the same time, Chen–Giga–Goto [7] established a level set approach for degenerate parabolic PDEs. In [2], Barles–Sonner–Souganidis rigorously connected the level set approach and the phase field theory for reaction-diffusion equations. The abstract method for front propagation was developed in [1, 3]. Roughly speaking, the abstract method is a tool that allows comparison of the sets $^{\pm}\Omega_t$ with the limiting set $\{u^{\varepsilon}(t, \cdot) > 0\}$ as $\varepsilon \to 0$.

2.1. **Weak solutions.** Due to the underlying geometry of Theorem 1.1, it is helpful to pass the notion of viscosity solutions of the PDE (2.2) to weak solutions of the level sets of the solution u. We use the notion of *generalized flows* for the mean curvature equation. The original definition of generalized flows is given in [3], see also [1]. We use the variation in [19] which is sufficient for proving the abstract method in [3].

Let $F(p, X)$ be given by

$$F(p, X) = -\mu \operatorname{tr}\left((I - \widehat{p} \otimes \widehat{p})X\right)$$

and the lower and upper semi-continuous envelopes of F be denoted by F_* and F^* respectively.

Definition 2.1. A family $(\Omega_t)_{t>0}$ of open (resp., closed) subsets of $\mathbb{R}^n$ is a generalized super-flow (resp., sub-flow) of the mean curvature equation (2.2) if for all $(t_0, x_0) \in (0, \infty) \times \mathbb{R}^n$, $h > 0$, and for all smooth functions $\varphi : (0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ such that

(i) (Boundedness) There exists $r > 0$ such that

$$\{(t, x) \in [t_0, t_0 + h] \times \mathbb{R}^n : \varphi(t, x) \geq 0\} \subset [t_0, t_0 + h] \times B(x_0, r),$$

(ii) (Strict subsolution) There exists $\delta = \delta(\varphi) > 0$ such that

$$\partial_t\varphi + F^*(\nabla\varphi, D^2\varphi) \leq -\delta \quad \text{in } [t_0, t_0 + h] \times \overline{B}(x_0, r),$$

$$(\text{resp., } \partial_t\varphi + F_*(\nabla\varphi, D^2\varphi) \geq \delta)$$

(iii) (Non-degeneracy)

$$\nabla\varphi \neq 0 \quad \text{in } \{(t,x) \in [t_0, t_0+h] \times \overline{B}(x_0,r) : \varphi(t,x) = 0\},$$

(iv) (Initial condition)

$$\{x \in \overline{B}(x_0,r) : \varphi(t_0,x) \geq 0\} \subset \Omega_{t_0}$$
$$(\text{resp., } \{x \in \overline{B}(x_0,r) : \varphi(t_0,x) \leq 0\} \subset \mathbb{R}^n \setminus \Omega_{t_0}),$$

then

$$\{x \in \bar{B}(x_0,r) : \varphi(t_0+h,x) > 0\} \subset \Omega_{t_0+h}$$
$$(\text{resp., } \{x \in \bar{B}(x_0,r) : \varphi(t_0+h,x) < 0\} \subset \mathbb{R}^n \setminus \Omega_{t_0+h}).$$

For the interested reader, we remark that $(\Omega_t)_{t\geq 0}$ is a generalized super-flow (sub-flow) of (2.2) if and only if $\mathbb{1}_{\Omega_t} - \mathbb{1}_{(\overline{\Omega}_t)^c}$ is a viscosity super(sub) solution of (2.2), see [3, Theorem 2.4]. For an introduction and background on viscosity solutions, see for example [8].

3. Preliminary results on the fractional Laplacian

In this section we recall a few basic properties of the operator $\mathcal{I}_n$ that will be used later on in the paper. First of all, by using that

$$\text{P. V.} \int_{|y|<R} \nabla u(x) \cdot y \, \frac{dy}{|y|^{n+1}} = 0,$$

for any $R > 0$, we can write

$$\mathcal{I}_n u(x) = \int_{|y|<R} (u(x+y) - u(x) - \nabla u(x) \cdot y) \, \frac{dy}{|y|^{n+1}} + \int_{|y|>R} (u(x+y) - u(x)) \, \frac{dy}{|y|^{n+1}}.$$

In particular if u is smooth and bounded, both integrals above are finite and we can bound $\mathcal{I}_n u$ as follows,

$$|\mathcal{I}_n u(x)| \leq C \left(\|D^2 u\|_\infty R + \frac{\|u\|_\infty}{R} \right),$$

where we used the following lemma whose proof is just a direct computations in polar coordinates.

Lemma 3.1. *There exists C_1, $C_2 > 0$ such that for any $R > 0$,*

$$\int_{\{|z|<R\}} \frac{dz}{|z|^{n-1}} = C_1 R \quad \text{and} \quad \int_{\{|z|>R\}} \frac{dz}{|z|^{n+1}} = \frac{C_2}{R}.$$

The next is an auxiliary lemma that allows us to view one-dimensional fractional Laplacians of functions defined over $\mathbb{R}$ equivalently as n-dimensional fractional Laplacians.

Lemma 3.2. *For a vector $e \in \mathbb{R}^n$ and a function $v \in C^{1,1}(\mathbb{R})$, let $v_e(x) = v(e \cdot x) : \mathbb{R}^n \to \mathbb{R}$. Then,*

$$\mathcal{I}_n[v_e](x) = |e| C_n \mathcal{I}_1[v](e \cdot x)$$

where

$$C_n = \int_{\mathbb{R}^{n-1}} \frac{1}{(|y|^2+1)^{\frac{n+1}{2}}} \, dy. \tag{3.1}$$

Consequently,

$$|e| C_n \mathcal{I}_1[v](\xi) = \text{P. V.} \int_{\mathbb{R}^n} (v(\xi + e \cdot z) - v(\xi)) \, \frac{dz}{|z|^{n+1}}, \quad \xi \in \mathbb{R}.$$

Proof. The case $e=0$ is trivial. Therefore, let us assume $e \neq 0$ and let $c := |e| > 0$. Begin by writing

$$\mathcal{I}_n[v_e](x) = \mathrm{P.V.} \int_{\mathbb{R}^n} (v(e\cdot x + e\cdot z) - v(e\cdot x)) \frac{dz}{|z|^{n+1}}.$$

We claim that it is enough to prove the result for $e = ce_1$, with $e_1 = (1, \dots, 0)$. Indeed, let T be a rotation matrix such that $Te = ce_1$ and apply the change of variables $Tz = y$ to obtain

$$\begin{aligned}\mathcal{I}_n[v_e](x) &= \mathrm{P.V.} \int_{\mathbb{R}^n} \left(v(e\cdot x + e\cdot T^{-1}y) - v(e\cdot x)\right) \frac{dy}{|T^{-1}y|^{n+1}}\\ &= \mathrm{P.V.} \int_{\mathbb{R}^n} (v(ce_1\cdot Tx + ce_1\cdot y) - v(ce_1\cdot Tx)) \frac{dy}{|y|^{n+1}} = \mathcal{I}_n[v_{ce_1}](Tx).\end{aligned}$$

If $\mathcal{I}_n[v_{ce_1}](x_0) = cC_n\mathcal{I}_1[v](ce_1\cdot x_0)$ for any $x_0 \in \mathbb{R}$, then we take $x_0 = Tx$ and notice that

$$\mathcal{I}_n[v_e](x) = \mathcal{I}_n[v_{ce_1}](Tx) = cC_n\mathcal{I}_1[v](ce_1\cdot Tx) = |e|C_n\mathcal{I}_1[v](e\cdot x).$$

Hence, the result holds.

It remains to prove the lemma for $e = ce_1$. Observe for $x = (x_1, x') \in \mathbb{R}\times\mathbb{R}^{n-1}$ that

$$\begin{aligned}\mathcal{I}_n[v_{ce_1}](x) &= \mathrm{P.V.} \int_{\mathbb{R}^n} (v(cx_1 + cz_1) - v(cx_1)) \frac{dz}{|z|^{n+1}}\\ &= \mathrm{P.V.} \int_{\mathbb{R}} (v(cx_1 + cz_1) - v(cx_1)) \left(\int_{\mathbb{R}^{n-1}} \frac{1}{|(z_1,z')|^{n+1}}\, dz'\right) dz_1.\end{aligned}$$

Since

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} \frac{1}{|(z_1,z')|^{n+1}}\, dz' &= \int_{\mathbb{R}^{n-1}} \frac{1}{(|z'|^2 + z_1^2)^{\frac{n+1}{2}}}\, dz'\\ &= \frac{1}{|z_1|^{n+1}} \int_{\mathbb{R}^{n-1}} \frac{1}{(|y|^2+1)^{\frac{n+1}{2}}}\, |z_1|^{n-1}\, dy = \frac{C_n}{|z_1|^2},\end{aligned}$$

we have

$$\begin{aligned}\mathcal{I}_n[v_{ce_1}](\xi) &= C_n\, \mathrm{P.V.} \int_{\mathbb{R}} (v(cx_1 + cz_1) - v(cx_1)) \frac{dz_1}{|z_1|^2}\\ &= cC_n\, \mathrm{P.V.} \int_{\mathbb{R}} (v(cx_1 + z_1) - v(x_1)) \frac{dz_1}{|z_1|^2}\\ &= cC_n\mathcal{I}_1[v](ce_1\cdot x).\end{aligned}$$

□

4. The phase transition, the corrector, and the auxiliary functions

In this section, we will introduce the phase transition ϕ and the corrector ψ. Along the way, we will also define the auxiliary functions a_ε and $\bar{a}_\varepsilon$ and exhibit their relationship with fractional Laplacians and the mean curvature equation, respectively.

4.1. **The phase transition ϕ.**
Let ϕ be the solution to (1.4). For convenience in the notation, let c_0 and α be given respectively by

$$c_0^{-1} = \int_{\mathbb{R}} [\dot{\phi}(\xi)]^2\, d\xi \quad \text{and} \quad \alpha = \frac{W''(0)}{C_n} \tag{4.1}$$

and let $H(\xi)$ be the Heaviside function and C_n is defined as in (3.1).

Lemma 4.1. *There is a unique solution $\phi \in C^{4,\beta}(\mathbb{R})$ of (1.4). Moreover, there exists a constant $C > 0$ such that*

$$\left|\phi(\xi) - H(\xi) + \frac{1}{\alpha\xi}\right| \le \frac{C}{|\xi|^2}, \quad \text{for } |\xi| \ge 1 \tag{4.2}$$

and

$$\frac{1}{C|\xi|^2} \le \dot\phi(\xi) \le \frac{C}{|\xi|^2}, \quad |\ddot\phi(\xi)| \le \frac{C}{|\xi|^2}, \quad |\dddot\phi(\xi)| \le \frac{C}{|\xi|^2}, \quad \text{for } |\xi| \ge 1. \tag{4.3}$$

Proof. Existence of a unique solution $\phi \in C^{2,\alpha}(\mathbb{R})$ of (1.4) is proven in [5], see Theorem 1.2 and Lemma 2.3. The regularity of W, (1.3), implies that ϕ is actually in $C^{4,\beta}(\mathbb{R})$, see [5, Lemma 2.3]. Estimate (4.2) and the estimate on $\dot\phi$ in (4.3) are proven in [16, Theorem 3.1]. Finally, the estimates on $\ddot\phi$ and $\dddot\phi$ in (4.3) are proven in [22, Lemma 3.1]. □

4.2. **The auxiliary functions a_ε and $\bar a_\varepsilon$.** Next, we will introduce two auxiliary functions that are necessary for our analysis. Let $d = d(t,x)$ be a given smooth function. Define the function $a_\varepsilon = a_\varepsilon(\xi; t, x)$ by

$$a_\varepsilon = \int_{\mathbb{R}^n} \left(\phi\left(\xi + \frac{d(t,x+\varepsilon z) - d(t,x)}{\varepsilon}\right) - \phi\left(\xi + \nabla d(t,x)\cdot z\right)\right)\frac{dz}{|z|^{n+1}}, \tag{4.4}$$

where $(\xi, t, x) \in \mathbb{R} \times [0,\infty) \times \mathbb{R}^n$. Notice that by Lemma 3.1 and by the regularity of ϕ and d the integral in (4.4) is well defined.

The corresponding function $\bar a_\varepsilon = \bar a_\varepsilon(t,x)$ is given by

$$\bar a_\varepsilon(t,x) = \frac{1}{\varepsilon|\ln\varepsilon|}\int_{\mathbb{R}} a_\varepsilon(\xi; t, x)\,\dot\phi(\xi)\,d\xi. \tag{4.5}$$

We have the following general estimate on $a_\varepsilon, \bar a_\varepsilon$. The proof is delayed until Section 8.

Lemma 4.2. *There exists $C > 0$ such that, for all $(\xi, t, x) \in \mathbb{R} \times (0,\infty) \times \mathbb{R}^n$,*

$$\|a_\varepsilon\|_{C^{3,1}_\xi(\mathbb{R})} \le C\varepsilon^{\frac12}, \tag{4.6}$$

$$\|\partial_t a_\varepsilon\|_{C^{3,1}_\xi(\mathbb{R})}, \|\nabla_x a_\varepsilon\|_{C^{3,1}_\xi(\mathbb{R})}, \|D^2_x a_\varepsilon\|_{C^{3,1}_\xi(\mathbb{R})} \le C\varepsilon^{\frac12} \tag{4.7}$$

$$|a_\varepsilon(\xi; t, x)|, \ |\dot a_\varepsilon(\xi; t, x)| \le \frac{C}{1+|\xi|}, \tag{4.8}$$

$$\begin{aligned} |\partial_t a_\varepsilon(\xi;t,x)|, |\nabla_x a_\varepsilon(\xi;t,x)|, \left|D^2_x a_\varepsilon(\xi;t,x)\right| &\le \frac{C}{1+|\xi|}, \\ |\partial_t \dot a_\varepsilon(\xi;t,x)|, |\nabla_x \dot a_\varepsilon(\xi;t,x)|, \left|D^2_x \dot a_\varepsilon(\xi;t,x)\right| &\le \frac{C}{1+|\xi|}. \end{aligned} \tag{4.9}$$

Consequently, for all $(t,x) \in (0,\infty) \times \mathbb{R}^n$,

$$|\bar a_\varepsilon(t,x)| \le \frac{C}{\varepsilon^{\frac12}|\ln\varepsilon|}, \tag{4.10}$$

and

$$|\partial_t \bar a_\varepsilon(t,x)|, |\nabla_x \bar a_\varepsilon(t,x)|, \left|D^2_x \bar a_\varepsilon(t,x)\right| \le \frac{C}{\varepsilon^{\frac12}|\ln\varepsilon|}. \tag{4.11}$$

We will be interested in a_ε and $\overline{a}_\varepsilon$ when d is the signed distance function to a front Γ_t. In this case, one of the main observations in [19] is that $\bar{a}_\varepsilon$ converges to the mean curvature of d in a neighborhood of Γ_t, see Lemma 4.4 below. However, we must take care because the signed distance function itself is not smooth everywhere. Throughout the paper, we will use the following smooth extension of the distance function away from Γ_t.

Definition 4.3 (Extension of the signed distance function). Let $\rho > 0$ be such that the signed distance function $\tilde{d}$ associated to a curve Γ_t is smooth in

$$Q_{2\rho} = \{(t,x) : |\tilde{d}(t,x)| \le 2\rho\}.$$

Consequently, $|\nabla \tilde{d}| = 1$ in $Q_{2\rho}$. Let $\eta(t,x)$ be a smooth, bounded function such that

$$\eta = 1 \text{ in } \{|\tilde{d}| < \rho\}, \quad \eta = 0 \text{ in } \{|\tilde{d}| > 2\rho\}, \quad 0 \le \eta \le 1.$$

We extend $\tilde{d}(t,x)$ in the set $\{|\tilde{d}| > \rho\}$ with the smooth bounded function $d(t,x)$ given by

$$d(t,x) = \begin{cases} \tilde{d}(t,x) & \text{in } \{|\tilde{d}(t,x)| \le \rho\} \\ \tilde{d}(t,x)\eta(t,x) + 2\rho(1-\eta(t,x)) & \text{in } \{\rho < \tilde{d}(t,x) < 2\rho\} \\ \tilde{d}(t,x)\eta(t,x) - 2\rho(1-\eta(t,x)) & \text{in } \{-2\rho < \tilde{d}(t,x) < -\rho\} \\ 2\rho & \text{in } \{\tilde{d}(t,x) \ge 2\rho\} \\ -2\rho & \text{in } \{\tilde{d}(t,x) \le -2\rho\} \end{cases}$$

Notice that in $\{\rho < \tilde{d} < 2\rho\}$ d satisfies

$$d = 2\rho + (\tilde{d} - 2\rho)\eta > 2\rho - \rho\eta \ge \rho,$$

and in $\{-2\rho < \tilde{d} < -\rho\}$ d satisfies

$$d = -2\rho + (\tilde{d} + 2\rho)\eta < -2\rho + \rho\eta \le -\rho.$$

Lemma 4.4 ([19], Lemma 4). *Let d be as in Definition 4.3. Then,*

$$\lim_{\varepsilon \to 0} \bar{a}_\varepsilon(t,x) = \frac{1}{2}\frac{|S^{n-2}|}{n-1}\Delta d(t,x) = \mu c_0^{-1} \operatorname{tr}\left((I - \widehat{\nabla d} \otimes \widehat{\nabla d})D^2 d\right)$$

uniformly in $(t,x) \in Q_\rho$ where $\mu > 0$ is in (1.6).

The proof of Lemma 4.4 is very technical. For the sake of completeness, we provide our own proof in Section 11. Note that, unlike [19], we utilize the asymptotics of ϕ in Lemma 4.1.

It is also important to notice that when $\xi = d(t,x)/\varepsilon$, morally, $a_\varepsilon(d/\varepsilon)$ is the difference between an n-dimensional and a 1-dimensional fractional Laplacian of $\phi(d/\varepsilon)$. This is seen in the following two lemmas.

Lemma 4.5 (Near the front). *Let d be as in Definition 4.3. If $|d(t,x)| \le \rho$, then*

$$a_\varepsilon\left(\frac{d(t,x)}{\varepsilon}; t, x\right) = \varepsilon \mathcal{I}_n\left[\phi\left(\frac{d(t,\cdot)}{\varepsilon}\right)\right](x) - C_n \mathcal{I}_1[\phi]\left(\frac{d(t,x)}{\varepsilon}\right).$$

Proof. First, we write $a_\varepsilon = a_\varepsilon\left(d(t,x)/\varepsilon; t, x\right)$ as

$$\begin{aligned} a_\varepsilon &= \int_{\mathbb{R}^n} \left(\phi\left(\frac{d(t,x+\varepsilon z)}{\varepsilon}\right) - \phi\left(\frac{d(t,x)}{\varepsilon} + \nabla d(t,x) \cdot z\right)\right) \frac{dz}{|z|^{n+1}} \\ &= \mathrm{P.V.} \int_{\mathbb{R}^n} \left(\phi\left(\frac{d(t,x+\varepsilon z)}{\varepsilon}\right) - \phi\left(\frac{d(t,x)}{\varepsilon}\right)\right) \frac{dz}{|z|^{n+1}} \end{aligned}$$

$$- \mathrm{P.V.}\int_{\mathbb{R}^n}\left(\phi\left(\frac{d(t,x)}{\varepsilon}+\nabla d(t,x)\cdot z\right)-\phi\left(\frac{d(t,x)}{\varepsilon}\right)\right)\frac{dz}{|z|^{n+1}}.$$

Since $e=\nabla d(t,x)$ is a unit vector when $|d(t,x)|\le\rho$, by applying Lemma 3.2 to the second integral and a change of variables in the first integral, we obtain

$$\begin{aligned}a_\varepsilon &= \varepsilon\,\mathrm{P.V.}\int_{\mathbb{R}^n}\left(\phi\left(\frac{d(t,x+z)}{\varepsilon}\right)-\phi\left(\frac{d(t,x)}{\varepsilon}\right)\right)\frac{dz}{|z|^{n+1}}-C_n\mathcal{I}_1[\phi]\left(\frac{d(t,x)}{\varepsilon}\right)\\ &= \varepsilon\mathcal{I}_n\left[\phi\left(\frac{d(t,\cdot)}{\varepsilon}\right)\right](x)-C_n\mathcal{I}_1[\phi]\left(\frac{d(t,x)}{\varepsilon}\right).\end{aligned}$$

□

Lemma 4.6 (Far from the front)**.** *Let d be as in Defintion 4.3. If $|d(t,x)|>\rho$, then there is a constant $C>0$ such that*

$$\left|a_\varepsilon\left(\frac{d(t,x)}{\varepsilon};t,x\right)-\left[\varepsilon\mathcal{I}_n\left[\phi\left(\frac{d(t,\cdot)}{\varepsilon}\right)\right](x)-C_n\mathcal{I}_1[\phi]\left(\frac{d(t,x)}{\varepsilon}\right)\right]\right|\le\frac{C\varepsilon}{\rho}.$$

The proof of Lemma 4.6 is delayed until Section 9.

4.3. **The corrector ψ.** The linearized operator $\mathcal{L}$ associated to (1.4) around ϕ is given by

$$\mathcal{L}[\psi]=-C_n\mathcal{I}_1[\psi]+W''(\phi)\psi. \tag{4.12}$$

In the constructions of barriers, we will need the corrector $\psi=\psi(\xi;t,x)$ that solves

$$\begin{cases}\mathcal{L}[\psi]=\dfrac{a_\varepsilon(\xi;t,x)}{\varepsilon|\ln\varepsilon|}+c_0\dot\phi(\xi)(\sigma-\bar a_\varepsilon(t,x))+\tilde\sigma\left(W''(\phi(\xi))-W''(0)\right), & \xi\in\mathbb{R}\\ \psi(\pm\infty;t,x)=0,\end{cases} \tag{4.13}$$

where $\sigma>0$ is a small positive constant and $\tilde\sigma>0$ is such that $\sigma=W''(0)\tilde\sigma$. See Section 5 for a formal derivation of (4.13).

To prove existence of the solution of (4.13) we will use the following result which is proven in [16].

Lemma 4.7. *Let $g\in H^{\frac12}(\mathbb{R})$ be such that*

$$\int_{\mathbb{R}} g(\xi)\dot\phi(\xi)\,d\xi=0.$$

Then, there exists a unique solution $\psi\in H^{\frac12}(\mathbb{R})$ such that $\int_{\mathbb{R}}\psi(\xi)\dot\phi(\xi)\,d\xi=0$ of

$$\mathcal{L}[\psi]=g\quad\text{in }\mathbb{R}.$$

Moreover, if $g\in C^{3,\beta}(\mathbb{R})$ with β as in (1.3), then $\psi\in C^{3,\beta}(\mathbb{R})$, and

$$\|\psi\|_{C^{3,\beta}(\mathbb{R})}\le C(\|g\|_{C^{3,\beta}(\mathbb{R})}+\|\psi\|_{L^\infty(\mathbb{R})}). \tag{4.14}$$

Proof. The proof of existence of a unique solution $\psi\in H^{\frac12}(\mathbb{R})$ satisfying $\int_{\mathbb{R}}\psi(\xi)\dot\phi(\xi)\,d\xi=0$ is contained in the proof of [16, Theorem 3.1]. If $g\in C^{3,\beta}(\mathbb{R})\cap H^{\frac12}(\mathbb{R})$, then for all $p\ge2$,

$$\|g\|_{L^p(\mathbb{R})}\le C$$

and the boundedness of ψ follows from [16, Corollary 5.16]. The $C^{1,\beta}$ regularity of ψ then is a consequence of [16, Proposition 5.17], if $g\in C^{0,\beta}(\mathbb{R})$. If in addition $g\in C^{3,\beta}(\mathbb{R})$, [5, Lemma 2.3] guarantees that ψ is actually in $C^{3,\beta}(\mathbb{R})$. □

Lemma 4.8. *Let*

$$g(\xi;t,x) := \frac{a_\varepsilon(\xi;t,x)}{\varepsilon\,|\ln\varepsilon|} + c_0\dot\phi(\xi)\,(\sigma - \bar a_\varepsilon(t,x)) + \tilde\sigma\left(W''(\phi(\xi)) - W''(0)\right). \tag{4.15}$$

Then,

$$\int_{\mathbb{R}} g(\xi;t,x)\dot\phi(\xi)\,d\xi = 0. \tag{4.16}$$

Moreover, $g \in H^{\frac12}_\xi(\mathbb{R}) \cap C^{3,\beta}_\xi(\mathbb{R})$, uniformly in $(t,x)\in(0,\infty)\times\mathbb{R}^n$.

Proof. Recalling that $\sigma = W''(0)\tilde\sigma$, we compute

$$\begin{aligned}\int_{\mathbb{R}} g(\xi;t,x)\dot\phi(\xi)\,d\xi &= \int_{\mathbb{R}}\left(\frac{1}{\varepsilon\,|\ln\varepsilon|}a_\varepsilon(\xi;t,x) - \dot\phi(\xi)\,c_0\bar a_\varepsilon(t,x)\right)\dot\phi(\xi)\,d\xi\\ &\quad+\int_{\mathbb{R}}\left(c_0\sigma\dot\phi(\xi) + \tilde\sigma W''(\phi(\xi)) - \sigma\right)\dot\phi(\xi)\,d\xi.\end{aligned}$$

Using the definitions of c_0 and $\bar a_\varepsilon$, (4.1) and (4.5) respectively, we get

$$\int_{\mathbb{R}}\left(\frac{1}{\varepsilon\,|\ln\varepsilon|}a_\varepsilon(\xi) - \dot\phi(\xi)\,c_0\bar a_\varepsilon(t,x)\right)\dot\phi(\xi)\,d\xi = \frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}} a_\varepsilon(\xi)\,\dot\phi(\xi)\,d\xi - \bar a_\varepsilon(t,x) = 0.$$

Then, we use that W' is periodic, $\phi(\infty)=1$ and $\phi(-\infty)=0$ to find that

$$\int_{\mathbb{R}}\tilde\sigma W''(\phi(\xi))\,\dot\phi(\xi)\,d\xi = \tilde\sigma\int_{\mathbb{R}}\frac{d}{d\xi}[W'(\phi(\xi))]\,d\xi = \tilde\sigma[W'(1)-W'(0)] = 0$$

and again the definition of c_0 to see that

$$\int_{\mathbb{R}}\left(c_0\sigma\dot\phi(\xi) - \sigma\right)\dot\phi(\xi)\,d\xi = c_0\sigma\int_{\mathbb{R}}[\dot\phi(\xi)]^2\,d\xi - \sigma = 0,$$

as desired. This proves (4.16).

Next, from (4.2), (4.3) and (4.8) we have that

$$W''(\phi(\xi)) - W''(0) = O(\phi(\xi)),\ \dot\phi(\xi),\ a_\varepsilon(\xi;t,x)\in H^1_\xi(\mathbb{R}),$$

which implies that $g\in H^{\frac12}_\xi(\mathbb{R})$, with $\|g\|_{H^{\frac12}_\xi(\mathbb{R})} \le C_\varepsilon$ for all $(t,x)\in(0,\infty)\times\mathbb{R}^n$. Moreover, from the regularity of ϕ and W and (4.6), it follows that $g\in C^{3,\beta}_\xi(\mathbb{R})$ with $\|g\|_{C^{3,\beta}_\xi(\mathbb{R})}\le C_\varepsilon$, for all $(t,x)\in(0,\infty)\times\mathbb{R}^n$. □

Theorem 4.9. *There is a unique solution $\psi=\psi(\xi;t,x)\in C^{3,\beta}(\mathbb{R})$ to (4.13) such that, for all $(\xi,t,x)\in\mathbb{R}\times(0,\infty)\times\mathbb{R}^n$*

$$\begin{aligned}&|\psi(\xi;t,x)|,\ |\dot\psi(\xi;t,x)|,\ |\ddot\psi(\xi;t,x)| \le \frac{C}{\varepsilon^{\frac12}|\ln\varepsilon|},\\ &|\partial_t\psi(\xi;t,x)|,\ |\nabla_x\psi(\xi;t,x)|,\ |D^2_x\psi(\xi;t,x)|,\ |\nabla_x\dot\psi(\xi;t,x)| \le \frac{C}{\varepsilon^{\frac12}|\ln\varepsilon|},\end{aligned} \tag{4.17}$$

and

$$|\psi(\xi;t,x)| \le \frac{C}{\varepsilon|\ln\varepsilon|(1+|\xi|)}, \tag{4.18}$$

for some $C>0$.

Proof. First, the existence of a unique solution $\psi \in H_\xi^{\frac{1}{2}}(\mathbb{R}) \cap C_\xi^{3,\beta}(\mathbb{R})$ to (4.13) satisfying $\int_{\mathbb{R}} \psi(\xi;t,x)\dot{\phi}(\xi)\,d\xi = 0$ follows from Lemmas 4.7 and 4.8.

Let us show (4.17) for ψ. Let g be defined as in (4.15). We first notice that by (4.6) and (4.10), we have that, for all $(\xi,t,x) \in \mathbb{R}\times(0,\infty)\times\mathbb{R}^n$

$$|g(\xi;t,x)| \le \frac{C}{\varepsilon^{\frac{1}{2}}|\ln\varepsilon|}. \tag{4.19}$$

Suppose by contradiction that there is a sequence $\varepsilon_k \to 0$ as $k\to\infty$ such that if ψ_k is solution to (4.13) with $\varepsilon = \varepsilon_k$, then for some $(t_k,x_k) \in (0,\infty)\times\mathbb{R}^n$,

$$\|\psi_k(\cdot;t_k,x_k)\|_{L^\infty(\mathbb{R})} \ge \frac{1}{\varepsilon_k^{\frac{1}{2}}|\ln\varepsilon_k|b_k} \tag{4.20}$$

with $0 < b_k \to 0$ as $k\to\infty$. Define the function $\tilde{\psi}_k(\xi) := \psi_k(\cdot;t_k,x_k)/\|\psi_k(\cdot;t_k,x_k)\|_{L^\infty(\mathbb{R})}$. Clearly, $\|\tilde{\psi}_k\|_{L^\infty(\mathbb{R})} = 1$, $\int_{\mathbb{R}} \tilde{\psi}_k(\xi)\dot{\phi}(\xi)\,d\xi = 0$ and $\tilde{\psi}_k$ solves

$$\mathcal{L}[\tilde{\psi}_k] = \tilde{g}_k \quad \text{in } \mathbb{R},$$

with

$$\tilde{g}_k(\xi) = \frac{g(\xi;t,x)}{\|\psi_k(\cdot;t_k,x_k)\|_{L^\infty(\mathbb{R})}}.$$

Notice that by (4.19) and (4.20), $\tilde{g}_k \to 0$ as $k\to\infty$ uniformly in $\mathbb{R}$. By (4.14) the sequence of functions ψ_k is uniformly bounded in $C^{3,\beta}(\mathbb{R})$, therefore, up to a subsequence, it converges in $C^3(\mathbb{R})$ to a function $\psi_\infty \in H_\xi^{\frac{1}{2}}(\mathbb{R}) \cap C_\xi^{3,\beta}(\mathbb{R})$ which is solution to

$$\mathcal{L}[\psi_\infty] = 0 \quad \text{in } \mathbb{R}.$$

Moreover, $\int_{\mathbb{R}} \tilde{\psi}_\infty(\xi)\dot{\phi}(\xi)\,d\xi = 0$. Indeed, by the uniform convergence of $\tilde{\psi}_k$ to ψ_∞ and using that $\phi(-\infty) = 0$ and $\phi(\infty) = 1$, as $k\to\infty$,

$$\begin{aligned}\left|\int_{\mathbb{R}} \tilde{\psi}_\infty(\xi)\dot{\phi}(\xi)\,d\xi\right| &= \left|\int_{\mathbb{R}} \tilde{\psi}_k(\xi)\dot{\phi}(\xi)\,d\xi - \int_{\mathbb{R}} \tilde{\psi}_\infty(\xi)\dot{\phi}(\xi)\,d\xi\right| \\ &\le \|\tilde{\psi}_k - \psi_\infty\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}} \dot{\phi}(\xi)\,d\xi \\ &= \|\tilde{\psi}_k - \psi_\infty\|_{L^\infty(\mathbb{R})} \to 0.\end{aligned}$$

The uniqueness of the solution guaranteed by Lemma 4.7 implies that $\psi_\infty \equiv 0$ which is in contradiction with $\|\psi_\infty\|_{L^\infty(\mathbb{R})} = 1$. This proves (4.17) for ψ. The estimates for $\dot{\psi}$ and $\ddot{\psi}$ then follow from (4.6), (4.14) and (4.17) for ψ, just proven.

Next, we establish (4.17) for the time/space derivatives of ψ. From (4.16) it follows that, for $i,j = 1,\dots,n$,

$$\int_{\mathbb{R}} \partial_t g(\xi;t,x)\dot{\phi}(\xi)\,d\xi = 0, \quad \int_{\mathbb{R}} \partial_{x_i} g(\xi;t,x)\dot{\phi}(\xi)\,d\xi = 0, \quad \int_{\mathbb{R}} \partial^2_{x_ix_j} g(\xi;t,x)\dot{\phi}(\xi)\,d\xi = 0.$$

Moreover, from (4.3), (4.9) and (4.11), the functions $\partial_t g$, $\partial_{x_i} g$ and $\partial^2_{x_ix_j} g$ belong to the space $H_\xi^{\frac{1}{2}}(\mathbb{R}) \cap C_\xi^{3,\beta}(\mathbb{R})$ uniformly in (t,x). Thus, it is easy to see that $\partial_t\psi$, $\partial_{x_i}\psi$ and $\partial^2_{x_ix_j}\psi$ exist and are the unique solutions in $H_\xi^{\frac{1}{2}}(\mathbb{R})$ such that $\int_{\mathbb{R}} \partial_t\psi(\xi;t,x)\dot{\phi}(\xi)\,d\xi = 0$, $\int_{\mathbb{R}} \partial_{x_i}\psi(\xi;t,x)\dot{\phi}(\xi)\,d\xi = 0$ and $\int_{\mathbb{R}} \partial^2_{x_ix_j}\psi(\xi;t,x)\dot{\phi}(\xi)\,d\xi = 0$, to

$$\mathcal{L}[\partial_t\psi] = \partial_t g, \quad \mathcal{L}[\partial_{x_i}\psi] = \partial^2_{x_ix_j} g, \quad \mathcal{L}[\partial^2_{x_ix_j}\psi] = \partial^2_{x_ix_j} g \quad \text{in } \mathbb{R},$$

respectively. Therefore, as above, from (4.7), the estimates in (4.17) for $\partial_t\psi$, $\nabla_x\psi$ and $D^2_x\psi$ follow. The estimate for $\nabla_x\dot{\psi}$ then follows from (4.14) with g and ψ replaced respectively by $\nabla_x g$ and $\nabla_x\psi$.

Finally, we prove estimate from (4.18). By (4.2), (4.3), (4.8) and (4.10), for $|\xi|\geq 1$,

$$|g(\xi;t,x)|\leq \frac{C}{\varepsilon|\ln\varepsilon||\xi|} \tag{4.21}$$

For $a>0$ let us denote $\phi_a(\xi)=\phi\left(\frac{\xi}{a}\right)$. Then, ϕ_a solves

$$C_n\mathcal{I}_1[\phi_a]=\frac{1}{a}W'(\phi_a)\quad\text{in }\mathbb{R}.$$

Therefore, recalling (4.1), that W is periodic and $W'(0)=0$, for $\xi\leq -1$, by (4.2) (note that $H(\xi)=0$),

$$\begin{aligned}\mathcal{L}[\phi_a](\xi)&=W''(\phi(\xi))\phi_a(\xi)-\frac{1}{a}W'(\phi_a(\xi))\\&=W''(0)\phi_a(\xi)-\frac{W''(0)}{a}\phi_a(x)+O(\phi\phi_a)+O(\phi_a^2)\\&=\alpha C_n\left(\phi_a(\xi)-\frac{1}{a}\phi_a(\xi)\right)+O\left(\frac{1}{\xi^2}\right)\\&=-C_n\frac{a-1}{\xi}+O\left(\frac{1}{\xi^2}\right).\end{aligned}$$

Choose $a=3$. Then, there exists $R_0>0$ such that,

$$\mathcal{L}[\phi_a](\xi)=-\frac{2}{\xi}+O\left(\frac{1}{\xi^2}\right)=\frac{2}{|\xi|}+O\left(\frac{1}{\xi^2}\right)\geq\frac{1}{|\xi|},\quad\text{for }\xi<-R_0. \tag{4.22}$$

Choose R_0 so large that for $\xi<-R_0$, by (4.2),

$$W''(\phi(\xi))\geq\alpha-C\phi(\xi)\geq\alpha-\frac{C}{|\xi|}\geq\frac{\alpha}{2}>0.$$

Then, the operator $\mathcal{L}$ satisfies the maximum principle in $(-\infty,-R_0)$. By (4.17) and the monotonicity of ϕ, for ε small enough,

$$\psi(\xi;t,x)\leq\frac{C\varepsilon^{\frac{1}{2}}}{\varepsilon|\ln\varepsilon|}\leq\frac{1}{\varepsilon|\ln\varepsilon|}\phi\left(-\frac{R_0}{3}\right)\leq\frac{\phi_a(\xi)}{\varepsilon|\ln\varepsilon|},\quad\text{for }\xi\geq -R_0. \tag{4.23}$$

Choose $K>1$ such that, if we denote $\hat{\phi}(\xi)=\frac{K\phi_a(\xi)}{\varepsilon|\ln\varepsilon|}$, then by (4.21) and (4.22),

$$\mathcal{L}[\hat{\phi}](\xi)\geq g(\xi;t,x)\quad\text{for }\xi<-R_0.$$

Since in addition by (4.23),

$$\psi(\xi;t,x)\leq\hat{\phi}(\xi)\quad\text{for }\xi\geq -R_0,$$

the maximum principle implies that

$$\psi(\xi;t,x)\leq\hat{\phi}(\xi)\leq\frac{C}{\varepsilon|\ln\varepsilon||\xi|}\quad\text{for }\xi<-R_0.$$

Comparing ψ with $-\hat{\phi}$ we also get

$$\psi(\xi;t,x)\geq-\frac{C}{\varepsilon|\ln\varepsilon||\xi|}\quad\text{for }\xi<-R_0.$$

Similarly, choosing $a < 0$, one can prove that

$$|\psi(\xi; t, x)| \le \frac{C}{\varepsilon |\ln \varepsilon| |\xi|} \quad \text{for } \xi > R_0.$$

Estimate (4.18) then follows. □

We conclude this section by stating the following estimate for the n- and 1-dimensional fractional Laplacians of ψ. The proof is in Section 10.

Lemma 4.10. *Let d be as in Definition 4.3. There is a constant $C > 0$ such that*

$$\left| \varepsilon \mathcal{I}_n \left[\psi \left(\frac{d(t,\cdot)}{\varepsilon}; t, \cdot \right) \right] (x) - C_n \mathcal{I}_1 [\psi(\cdot; t, x)] \left(\frac{d(t,x)}{\varepsilon} \right) \right| \le \frac{C}{|\ln \varepsilon|}.$$

for any $(t, x) \in (0, \infty) \times \mathbb{R}^n$.

5. Heuristics

5.1. **Ansatz for motion by mean curvature.** We believe it is helpful to view the heuristical derivation of the evolution of the fronts Γ_t^i by mean curvature in Theorem 1.1. For simplicity, we consider the case $N = 2$.

For the following formal computations, assume that the signed distance function $d_i(t, x)$ associated to Γ_t^i is smooth and that $|\nabla d_i| = 1$. Moreover, we assume that there is a positive, uniform distance ρ between Γ_t^1 and Γ_t^2.

The ansatz for the solution to the reaction-diffusion equation (1.1) is given by

$$u^\varepsilon(t, x) \simeq \phi\left(\frac{d_1(t,x)}{\varepsilon} \right) + \phi\left(\frac{d_2(t,x)}{\varepsilon} \right). \tag{5.1}$$

Plugging the ansatz into (1.1), the left-hand side gives

$$\varepsilon \partial_t u^\varepsilon \simeq \dot\phi\left(\frac{d_1}{\varepsilon} \right) \partial_t d_1 + \dot\phi\left(\frac{d_2}{\varepsilon} \right) \partial_t d_2. \tag{5.2}$$

Up to dividing by $\varepsilon |\ln \varepsilon|$, we use the equation for ϕ (see (1.4)) and estimates on a_ε (see Lemma 4.5 and Lemma 4.6) to write the right-hand side of (1.1) for the ansatz as

(5.3)

$$\begin{aligned}
&\varepsilon \mathcal{I}_n[u^\varepsilon] - W'(u^\varepsilon) \\
&\quad \simeq \varepsilon \mathcal{I}_n \left[\phi\left(\frac{d_1}{\varepsilon} \right) \right] + \varepsilon \mathcal{I}_n \left[\phi\left(\frac{d_2}{\varepsilon} \right) \right] - W'\left(\phi\left(\frac{d_1}{\varepsilon} \right) + \phi\left(\frac{d_2}{\varepsilon} \right) \right) \\
&\quad = \left(\varepsilon \mathcal{I}_n \left[\phi\left(\frac{d_1}{\varepsilon} \right) \right] - C_n \mathcal{I}_1[\phi] \left(\frac{d_1}{\varepsilon} \right) \right) + \left(\varepsilon \mathcal{I}_n \left[\phi\left(\frac{d_2}{\varepsilon} \right) \right] - C_n \mathcal{I}_1[\phi] \left(\frac{d_2}{\varepsilon} \right) \right) \\
&\qquad + C_n \mathcal{I}_1[\phi] \left(\frac{d_1}{\varepsilon} \right) + C_n \mathcal{I}_1[\phi] \left(\frac{d_2}{\varepsilon} \right) - W'\left(\phi\left(\frac{d_1}{\varepsilon} \right) + \phi\left(\frac{d_2}{\varepsilon} \right) \right) \\
&\quad \simeq a_\varepsilon \left(\frac{d_1}{\varepsilon} \right) + a_\varepsilon \left(\frac{d_2}{\varepsilon} \right) + W'\left(\phi\left(\frac{d_1}{\varepsilon} \right) \right) + W'\left(\phi\left(\frac{d_2}{\varepsilon} \right) \right) - W'\left(\phi\left(\frac{d_1}{\varepsilon} \right) + \phi\left(\frac{d_2}{\varepsilon} \right) \right).
\end{aligned}$$

Freeze a point (t, x) near the front Γ_t^1. Let $\xi = d_1(t, x)/\varepsilon$ and assume separation of scales. That is, assume that ξ and (t, x) are unrelated. In this regard, let $\eta = |d_2(t, x)| \ge \rho$, so that η^{-1} is bounded. Since the ansatz u^ε is a solution to (1.1), we can multiply the equation by by $\dot\phi(\xi)$ and integrate over $\xi \in \mathbb{R}$ to write

$$\int_{\mathbb{R}} \varepsilon \partial_t u^\varepsilon \dot\phi \, d\xi \simeq \frac{1}{\varepsilon |\ln \varepsilon|} \int_{\mathbb{R}} \left(\varepsilon \mathcal{I}_n u^\varepsilon - W'(u^\varepsilon) \right) \dot\phi(\xi) \, d\xi. \tag{5.4}$$

For convenience, we will consider the left and right-hand sides separately again.

First, the left-hand side of (5.4) with (5.2) gives

$$\begin{aligned}\int_{\mathbb{R}} \varepsilon \partial_t u^\varepsilon \, \dot{\phi}(\xi)\, d\xi &\simeq \partial_t d_1(t,x) \int_{\mathbb{R}} [\dot{\phi}\,(\xi)]^2 \, d\xi + \dot{\phi}\left(\frac{\eta}{\varepsilon}\right) \partial_t d_2(t,x) \int_{\mathbb{R}} \dot{\phi}(\xi)\, d\xi \\ &\simeq c_0^{-1} \partial_t d_1(t,x) + \frac{C\varepsilon^2}{\eta^2} \partial_t d_2(t,x) \\ &\simeq c_0^{-1} \partial_t d_1(t,x),\end{aligned}$$

where we used (4.1), (1.4), and the asymptotics on $\dot{\phi}$ (see (4.3)).

Then, we look at the right-hand side of (5.4) with (5.3). First, using that $\phi(-\infty) = 0$, $\phi(\infty) = 1$ and that W is periodic, we have

$$\frac{1}{\varepsilon\,|\ln \varepsilon|} \int_{\mathbb{R}} W'\left(\phi(\xi)\right) \dot{\phi}(\xi)\, d\xi = \frac{1}{\varepsilon\,|\ln \varepsilon|} \int_{\mathbb{R}} \frac{d}{d\xi} [W\left(\phi\,(\xi)\right)]\, d\xi = 0.$$

Next, we use the asymptotics and properties of ϕ (see (1.4), (4.2)) and that $W'(0) = 0$ to Taylor expand W' around the origin and estimate

$$\begin{aligned}\frac{1}{\varepsilon\,|\ln \varepsilon|} W'\left(\phi\left(\frac{\eta}{\varepsilon}\right)\right) \int_{\mathbb{R}} \dot{\phi}(\xi)\, d\xi &= \frac{1}{\varepsilon\,|\ln \varepsilon|} W'\left(\phi\left(\frac{\eta}{\varepsilon}\right) - H\left(\frac{\eta}{\varepsilon}\right)\right) \\ &\simeq \frac{1}{\varepsilon\,|\ln \varepsilon|} \left[W'(0) + W''(0)\left(\phi\left(\frac{\eta}{\varepsilon}\right) - H\left(\frac{\eta}{\varepsilon}\right)\right)\right] \\ &\simeq 0 + \frac{1}{\varepsilon\,|\ln \varepsilon|} \frac{C\varepsilon}{\eta} \simeq 0.\end{aligned}$$

For the remaining W' term, we Taylor expand around $\phi(\xi)$ and use similar estimates to obtain

$$\begin{aligned}&\frac{1}{\varepsilon\,|\ln \varepsilon|} \int_{\mathbb{R}} W'\left(\phi\,(\xi) + \phi\left(\frac{\eta}{\varepsilon}\right)\right) \dot{\phi}(\xi)\, d\xi \\ &\quad = \frac{1}{\varepsilon\,|\ln \varepsilon|} \int_{\mathbb{R}} W'\left(\phi\,(\xi) + \phi\left(\frac{\eta}{\varepsilon}\right) - H\left(\frac{\eta}{\varepsilon}\right)\right) \dot{\phi}(\xi)\, d\xi \\ &\quad \simeq \frac{1}{\varepsilon\,|\ln \varepsilon|} \int_{\mathbb{R}} \left[W'\left(\phi\,(\xi)\right) + W''\left(\phi\,(\xi)\right)\left(\phi\left(\frac{\eta}{\varepsilon}\right) - H\left(\frac{\eta}{\varepsilon}\right)\right)\right] \dot{\phi}(\xi)\, d\xi \\ &\quad = \frac{1}{\varepsilon\,|\ln \varepsilon|} \int_{\mathbb{R}} W'\left(\phi\,(\xi)\right) \dot{\phi}(\xi)\, d\xi + \frac{1}{\varepsilon\,|\ln \varepsilon|} \left(\phi\left(\frac{\eta}{\varepsilon}\right) - H\left(\frac{\eta}{\varepsilon}\right)\right) \int_{\mathbb{R}} W''\left(\phi\,(\xi)\right) \dot{\phi}(\xi)\, d\xi \\ &\quad \simeq 0 + \frac{1}{\varepsilon\,|\ln \varepsilon|} \frac{C\varepsilon}{\eta} \int_{\mathbb{R}} \frac{d}{d\xi} [W'\left(\phi\,(\xi)\right)]\, d\xi = 0.\end{aligned}$$

Lastly, for the nonlocal terms, by (4.8),

$$\frac{1}{\varepsilon\,|\ln \varepsilon|} a_\varepsilon\left(\frac{\eta}{\varepsilon}\right) \int_{\mathbb{R}} \dot{\phi}(\xi)\, d\xi \simeq 0$$

and by Lemma 4.4,

$$\begin{aligned}\frac{1}{\varepsilon\,|\ln \varepsilon|} \int_{\mathbb{R}} a_\varepsilon\,(\xi)\, \dot{\phi}(\xi)\, d\xi &= \bar{a}_\varepsilon(t,x) \\ &\simeq c_0^{-1} \mu \operatorname{tr}\left((I - \widehat{\nabla d_1(t,x)} \otimes \widehat{\nabla d_1(t,x)}) D^2 d_1(t,x)\right).\end{aligned}$$

Combing all these pieces, (5.4) for the ansatz gives

$$c_0^{-1} \partial_t d_1(t,x) \simeq \mu c_0^{-1} \operatorname{tr}\left((I - \widehat{\nabla d_1(t,x)} \otimes \widehat{\nabla d_1(t,x)}) D^2 d_1(t,x)\right).$$

The computation for (t,x) frozen near Γ_t^2 is similar. We conclude that the fronts move according to their mean curvature:

$$\begin{cases} \partial_t d_1(t,x) \simeq \mu \operatorname{tr}\left((I - \widehat{\nabla d_1(t,x)} \otimes \widehat{\nabla d_1(t,x)})D^2 d_1(t,x)\right) & \text{near } \Gamma_t^1 \\ \partial_t d_2(t,x) \simeq \mu \operatorname{tr}\left((I - \widehat{\nabla d_2(t,x)} \otimes \widehat{\nabla d_2(t,x)})D^2 d_2(t,x)\right) & \text{near } \Gamma_t^2. \end{cases}$$

5.2. **Ansatz for corrector.** One of the key ingredients in proving Theorem 1.1 is the construction of strict subsolutions (supersolutions), denoted by $v^\varepsilon = v^\varepsilon(t,x)$. For this, it is necessary to add a small corrector ψ to the ansatz in (5.1). In order to showcase the equation for ψ, we will consider the simplest case in which $N = 1$ and assume that $d(t,x) = d_1(t,x)$ is smooth with $|\nabla d| = 1$ and satisfies

$$\partial_t d = \mu \Delta d - c_0 \sigma \simeq c_0 \bar{a}_\varepsilon(t,x) - c_0 \sigma. \tag{5.5}$$

To find the corrector ψ for the barrier, we consider the ansatz

$$v^\varepsilon(t,x) \simeq \phi\left(\frac{d(t,x)}{\varepsilon}\right) + \varepsilon\,|\ln \varepsilon|\,\psi\left(\frac{d(t,x)}{\varepsilon}\right) - \varepsilon\,|\ln\varepsilon|\,\tilde{\sigma},$$

where the function ψ is to be determined and $\tilde{\sigma} > 0$ is a small, given constant. Assume for now that ψ is smooth and bounded with bounded derivative.

Since v^ε is a subsolution to (1.1), then heuristically, there is a $\sigma > 0$ such that

$$\varepsilon \partial_t v^\varepsilon = \frac{1}{\varepsilon\,|\ln\varepsilon|}(\varepsilon \mathcal{I}_n v^\varepsilon - W'(v^\varepsilon)) - \sigma. \tag{5.6}$$

Plugging the ansatz into (5.6), the left-hand side gives

$$\varepsilon \partial_t v^\varepsilon \simeq \dot{\phi}\left(\frac{d}{\varepsilon}\right)\partial_t d + \varepsilon\,|\ln\varepsilon|\,\dot{\psi}\left(\frac{d}{\varepsilon}\right)\partial_t d \simeq \dot{\phi}\left(\frac{d}{\varepsilon}\right)\partial_t d, \tag{5.7}$$

where we use that $\dot{\psi}$ and $\partial_t d$ are bounded. Next, we look at the right-hand side of (5.6) for the ansatz. First, we use the equation for ϕ (see (1.4)) and estimates on a_ε (see Lemmas 4.5, 4.6, 4.10) to find that

$$\begin{aligned} \frac{\varepsilon}{\varepsilon\,|\ln\varepsilon|}\mathcal{I}_n[v^\varepsilon] &\simeq \frac{\varepsilon}{\varepsilon\,|\ln\varepsilon|}\mathcal{I}_n\left[\phi\left(\frac{d}{\varepsilon}\right)\right] + \varepsilon\mathcal{I}_n\left[\psi\left(\frac{d}{\varepsilon}\right)\right] \\ &= \frac{1}{\varepsilon\,|\ln\varepsilon|}\left(\varepsilon\mathcal{I}_n\left[\phi\left(\frac{d}{\varepsilon}\right)\right] - C_n\mathcal{I}_1[\phi]\left(\frac{d}{\varepsilon}\right)\right) + \frac{1}{\varepsilon\,|\ln\varepsilon|}C_n\mathcal{I}_1[\phi]\left(\frac{d}{\varepsilon}\right) \\ &\quad + \left(\varepsilon\mathcal{I}_n\left[\psi\left(\frac{d}{\varepsilon}\right)\right] - C_n\mathcal{I}_1[\psi]\left(\frac{d}{\varepsilon}\right)\right) + C_n\mathcal{I}_1[\psi]\left(\frac{d}{\varepsilon}\right) \\ &\simeq \frac{1}{\varepsilon\,|\ln\varepsilon|}a_\varepsilon\left(\frac{d}{\varepsilon}\right) + \frac{1}{\varepsilon\,|\ln\varepsilon|}W'\left(\phi\left(\frac{d}{\varepsilon}\right)\right) + C_n\mathcal{I}_1[\psi]\left(\frac{d}{\varepsilon}\right). \end{aligned} \tag{5.8}$$

On the other hand, we do a Taylor expansion for W' around $\phi(d/\varepsilon)$ to estimate

$$\begin{aligned} \frac{1}{\varepsilon\,|\ln\varepsilon|}W'(v^\varepsilon) &\simeq \frac{1}{\varepsilon\,|\ln\varepsilon|}\left[W'\left(\phi\left(\frac{d}{\varepsilon}\right)\right) + W''\left(\phi\left(\frac{d)}{\varepsilon}\right)\right)\left(v^\varepsilon - \phi\left(\frac{d}{\varepsilon}\right)\right)\right] \\ &\simeq \frac{1}{\varepsilon\,|\ln\varepsilon|}\left[W'\left(\phi\left(\frac{d}{\varepsilon}\right)\right) + W''\left(\phi\left(\frac{d}{\varepsilon}\right)\right)\left(\varepsilon\,|\ln\varepsilon|\,\psi\left(\frac{d}{\varepsilon}\right) - \varepsilon\,|\ln\varepsilon|\,\tilde{\sigma}\right)\right]. \end{aligned} \tag{5.9}$$

Equating (5.7) with (5.8) and (5.9), the equation for the ansatz gives

$$\begin{aligned}\dot\phi\left(\frac{d}{\varepsilon}\right)\partial_t d \simeq{}& \frac{1}{\varepsilon\,|\ln\varepsilon|}a_\varepsilon\left(\frac{d}{\varepsilon}\right)+C_n\mathcal{I}_1[\psi]\left(\frac{d}{\varepsilon}\right)\\ &-W''\left(\phi\left(\frac{d}{\varepsilon}\right)\right)\psi\left(\frac{d}{\varepsilon}\right)+\tilde\sigma W''\left(\phi\left(\frac{d}{\varepsilon}\right)\right)-\sigma.\end{aligned}\tag{5.10}$$

Rearranging and using (5.5), we have

$$\begin{aligned}&-C_n\mathcal{I}_1[\psi]\left(\frac{d}{\varepsilon}\right)+W''\left(\phi\left(\frac{d}{\varepsilon}\right)\right)\psi\left(\frac{d}{\varepsilon}\right)\\ &\quad\simeq\frac{1}{\varepsilon\,|\ln\varepsilon|}a_\varepsilon\left(\frac{d}{\varepsilon}\right)-\dot\phi\left(\frac{d}{\varepsilon}\right)\partial_t d+\tilde\sigma W''\left(\phi\left(\frac{d}{\varepsilon}\right)\right)-\sigma\\ &\quad\simeq\frac{1}{\varepsilon\,|\ln\varepsilon|}a_\varepsilon\left(\frac{d}{\varepsilon}\right)-\dot\phi\left(\frac{d}{\varepsilon}\right)c_0\bar a_\varepsilon+c_0\sigma\dot\phi\left(\frac{d}{\varepsilon}\right)+\tilde\sigma W''\left(\phi\left(\frac{d}{\varepsilon}\right)\right)-\sigma.\end{aligned}$$

We let ψ be the solution to this equation. In particular, let $\mathcal{L}$ be the linearized operator in (4.12). Then, that corrector ψ satisfies the equation

$$\begin{aligned}\mathcal{L}[\psi]\left(\frac{d(t,x)}{\varepsilon}\right)={}&\frac{1}{\varepsilon\,|\ln\varepsilon|}a_\varepsilon\left(\frac{d(t,x)}{\varepsilon}\right)-\dot\phi\left(\frac{d(t,x)}{\varepsilon}\right)c_0\bar a_\varepsilon(t,x)\\ &+c_0\sigma\dot\phi\left(\frac{d(t,x)}{\varepsilon}\right)+\tilde\sigma W''\left(\phi\left(\frac{d(t,x)}{\varepsilon}\right)\right)-\sigma,\end{aligned}\tag{5.11}$$

as desired. See (4.13) with $\sigma = W''(0)\tilde\sigma$.

In order to check the validity equation (5.11), at least formally, we freeze a point (t, x) near Γ^1_t. Let $\xi = d(t, x)/\varepsilon$ and assume separation of scales. We multiply both sides of (5.11) by $\dot\phi(\xi)$ and integrate over $\mathbb{R}$ to write

$$\begin{aligned}\int_{\mathbb{R}}\mathcal{L}[\psi]\,(\xi)\,\dot\phi(\xi)\,d\xi={}&\int_{\mathbb{R}}\left(\frac{1}{\varepsilon\,|\ln\varepsilon|}a_\varepsilon\,(\xi)-\dot\phi\,(\xi)\,c_0\bar a_\varepsilon(t,x)\right)\dot\phi(\xi)\,d\xi\\ &+\int_{\mathbb{R}}\Big(c_0\sigma\dot\phi\,(\xi)+\tilde\sigma W''\,(\phi\,(\xi))-\sigma\Big)\,\dot\phi(\xi)\,d\xi.\end{aligned}$$

Since $\mathcal{I}_1$ is self-adjoint and ϕ satisfies (1.4), the left-hand side of the equation gives

$$\begin{aligned}\int_{\mathbb{R}}\mathcal{L}[\psi]\dot\phi\,d\xi&=\int_{\mathbb{R}}\Big(-C_n\mathcal{I}_1[\dot\phi]+W''\,(\phi)\,\dot\phi\Big)\,\psi\,d\xi\\ &=\int_{\mathbb{R}}\frac{d}{d\xi}\left(-C_n\mathcal{I}_1[\phi]+W'\,(\phi)\right)\psi\,d\xi=0.\end{aligned}\tag{5.12}$$

On the other hand, the right-hand side is also zero by Lemma 4.8.

Remark 5.1. Notice that ψ depends on the signed distance function $d(t, x)$. Hence, when $N > 1$, we have a finite sequence of correctors, denoted by $\psi_1, \dots, \psi_N$, depending on the signed distance function $d_i(t, x)$ to the front Γ^i_t, $i = 1, \dots, N$.

Remark 5.2. To see that $\sigma = W''(0)\tilde\sigma$, assume that $d(t, x) << -1$ and $\psi \equiv 0$. Then, $(t, x) \in {}^-\Omega^1_t$ is far from the front Γ^1_t. By Lemmas 4.1 and 4.6 (and Lemma 10.2), in (5.10),

$$0 \simeq 0 + \tilde\sigma W''\,(0) - \sigma.$$

6. Construction of barriers

We now construct the strict subsolutions (supersolutions) to (1.1) needed for the proof of Theorem 1.1. In particular, the barriers will be used to prove that a sequence of sets are generalized super(sub)-flows. We will focus on the construction of subsolutions as the construction of supersolutions is analogous.

Fix $(t_0, x_0) \in [0,\infty) \times \mathbb{R}^n$ and $h > 0$. Let $\varphi_i(t,x)$, $i = 1,\dots,N$, be smooth functions satisfying (i),(ii),(iii) in Definition 2.1 for $(t,x) \in [t_0, t_0+h] \times B(x_0, r_i)$. Moreover, assume

$$\{(t,x) : \varphi_{i+1}(t,x) > 0\} \subset\subset \{(t,x) : \varphi_i(t,x) > 0\} \quad \text{for } i = 1,\dots,N-1. \tag{6.1}$$

Let $\tilde{d}_i(t,x)$ be the signed distance function associated to the set $\{x \in \mathbb{R}^n : \varphi_i(t,x) > 0\}$, which we know to be bounded by (i) in Definition 2.1. Then, we can denote the zero level set of φ_i by $\Gamma_t^i = \{x \in \mathbb{R}^n : \tilde{d}_i(t,x) = 0\}$. As a consequence of (iii) in Definition 2.1, there is a $\rho > 0$ such that $\tilde{d}_i(t,x)$ is smooth in the set

$$Q_{2\rho}^i = \{(t,x) : |\tilde{d}_i(t,x)| \le 2\rho\}$$

and $|\nabla \tilde{d}_i| = 1$ in $Q_{2\rho}^i$. Moreover, by (6.1), and perhaps making ρ smaller, we can assume that $Q_\rho^i \cap Q_\rho^j = \emptyset$ for $i \neq j$. Let d_i be the smooth, bounded extension of $\tilde{d}_i$ outside of Q_ρ^i as defined in Definition 4.3.

Since for $x \in \Gamma_t^i$, we have $\partial_t d_i(t,x) = \partial_t \varphi_i(t,x)$ and $\nabla d_i(t,x) = \nabla \varphi_i(t,x)/|\nabla \varphi_i(t,x)|$, as a consequence of (ii) in Definition 2.1, for $\sigma > 0$ sufficiently small,

$$\partial_t d_i \le \mu \operatorname{tr}\left((I - \widehat{\nabla d_i} \otimes \widehat{\nabla d_i}) D^2 d_i\right) - c_0 \sigma = \mu \Delta d_i - c_0 \sigma \quad \text{in } Q_\rho^i. \tag{6.2}$$

Let $\tilde{\sigma} > 0$ be such that $\sigma = W''(0)\tilde{\sigma}$. Then, we define the smooth barrier $v^\varepsilon(t,x)$ by

$$v^\varepsilon(t,x) = \sum_{i=1}^N \phi\left(\frac{d_i(t,x) - \tilde{\sigma}}{\varepsilon}\right) + \varepsilon |\ln \varepsilon| \sum_{i=1}^N \psi_i\left(\frac{d_i(t,x) - \tilde{\sigma}}{\varepsilon}; t, x\right) - \tilde{\sigma}\varepsilon |\ln \varepsilon|. \tag{6.3}$$

Lemma 6.1. *For sufficiently small ε, v^ε is subsolution to*

$$\varepsilon \partial_t v^\varepsilon - \frac{1}{\varepsilon |\ln \varepsilon|}\left(\varepsilon \mathcal{I}_n[v^\varepsilon] - W'(v^\varepsilon)\right) < -\frac{\sigma}{2} \quad \textit{in } [t_0, t_0+h] \times \mathbb{R}^n. \tag{6.4}$$

Moreover, there is a constant $C > 0$ such that, for $\varepsilon > 0$ sufficiently small,

$$N - 2\tilde{\sigma}\varepsilon |\ln \varepsilon| \le v^\varepsilon(t,x) \le N - \frac{\tilde{\sigma}}{2}\varepsilon |\ln \varepsilon| \quad \textit{in } \left\{(t,x) : d_N(t,x) - \tilde{\sigma} \ge \frac{C}{\tilde{\sigma}|\ln \varepsilon|}\right\}. \tag{6.5}$$

Proof. We will break the proof into four main steps. First, we estimate the equation for $v^\varepsilon(t,x)$ for any (t,x). Then, we will show that $v^\varepsilon(t,x)$ satisfies (6.4) when (t,x) is near a single front $\Gamma_t^{i_0}$ and then when (t,x) is far from all fronts Γ_t^i, $i = 1,\dots,N$. Lastly, we prove the estimate in (6.5).

For convenience, we use the following notation throughout the proof:

$$\begin{aligned}
\phi_i &:= \phi\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right)\\
\psi_i &:= \psi_i\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon};t,x\right)\\
\tilde{\phi}_i &:= \phi\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right)-H\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right)\\
a_\varepsilon^i &:= a_\varepsilon\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon};t,x\right)\\
\bar{a}_\varepsilon^i &:= \bar{a}_\varepsilon(t,x) \quad \text{corresponding to } a_\varepsilon^i\\
b_\varepsilon^i &:= \varepsilon\mathcal{I}_n\left[\phi\left(\frac{d_i(t,\cdot)-\tilde{\sigma}}{\varepsilon}\right)\right](x)-C_n\mathcal{I}_1[\phi]\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right).
\end{aligned} \tag{6.6}$$

We note that it will be important for the reader to remember the dependence of ψ_i and a_ε on the variables t,x and $\xi = d_i(t,x)/\varepsilon$ when taking derivatives in t,x.

Step 1. Computation for $v^\varepsilon(t,x)$ in (1.1) for an arbitrary $(t,x)\in[t_0,t_0+h]\times\mathbb{R}^n$.

First, the time derivative of v^ε at (t,x) is given by

$$\varepsilon\partial_t v^\varepsilon(t,x)=\sum_{i=1}^N \dot{\phi}_i\,\partial_t d_i(t,x)+\varepsilon\,|\ln\varepsilon|\sum_{i=1}^N\left[\dot{\psi}_i\partial_t d_i(t,x)+\varepsilon\partial_t\psi_i\right].$$

By (4.17) for ψ and $\partial_t\psi_i$,

$$\varepsilon\partial_t v^\varepsilon=\sum_{i=1}^N\dot{\phi}_i\,\partial_t d_i(t,x)+O(\varepsilon^{\frac{1}{2}}). \tag{6.7}$$

Next, we consider the nonlocal term. For each $i=1,\dots,N$, we use that ϕ satisfies (1.4) to find

$$\varepsilon\mathcal{I}_n[\phi_i](x)=\varepsilon\mathcal{I}_n[\phi_i](x)-C_n\mathcal{I}_1[\phi]\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right)+W'(\phi_i).$$

Also, using that ψ satisfies (4.13) and Lemma 4.10, we find that

$$\begin{aligned}
\varepsilon\mathcal{I}_n[\psi_i](x)&=\varepsilon\mathcal{I}_n[\psi_i](x)-C_n\mathcal{I}_1[\psi_i]\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right)-\mathcal{L}[\psi]\left(\frac{d_i(t,\cdot)-\tilde{\sigma}}{\varepsilon}\right)+W''(\phi_i)\,\psi_i\\
&=O\left(|\ln\varepsilon|^{-1}\right)-\frac{1}{\varepsilon\,|\ln\varepsilon|}a_\varepsilon^i+\dot{\phi}_i c_0\left(\bar{a}_\varepsilon^i-\sigma\right)-\tilde{\sigma}\left(W''(\phi_i)-W''(0)\right)+W''(\phi_i)\,\psi_i.
\end{aligned}$$

Therefore, the 1/2-Laplacian of v^ε can be written as

$$\begin{aligned}
\varepsilon\mathcal{I}_n[v^\varepsilon](x)=&\sum_{i=1}^N\left[\varepsilon\mathcal{I}_n[\phi_i](x)-C_n\mathcal{I}_1[\phi]\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right)+W'(\phi_i)\right]\\
&+\varepsilon\,|\ln\varepsilon|\sum_{i=1}^N\Bigg[O\left(|\ln\varepsilon|^{-1}\right)-\frac{1}{\varepsilon\,|\ln\varepsilon|}a_\varepsilon^i+\dot{\phi}_i c_0\left(\bar{a}_\varepsilon^i-\sigma\right)\\
&\qquad-\tilde{\sigma}\left(W''(\phi_i)-W''(0)\right)+W''(\phi_i)\,\psi_i\Bigg].
\end{aligned}$$

Recall the definitions of $\tilde{\phi}_i$ and b_ε^i introduced in (6.6). Since W is periodic, we have that $W'(\phi_i) = W'(\tilde{\phi}_i)$ and similarly $W''(\phi_i) = W''(\tilde{\phi}_i)$. Using this, rearranging, and utilizing the notation b_ε^i, we can equivalently write

$$\begin{aligned}\varepsilon \mathcal{I}_n[v^\varepsilon](x) &= \sum_{i=1}^N \Big[(b_\varepsilon^i - a_\varepsilon^i) + W'(\tilde{\phi}_i)\Big] + O(\varepsilon) \\ &\quad + \varepsilon\,|\ln \varepsilon| \sum_{i=1}^N \Big[+W''(\tilde{\phi}_i)\psi_i + \dot{\phi}_i c_0 \left(\bar{a}_\varepsilon^i - \sigma\right) - \tilde{\sigma}\Big(W''(\tilde{\phi}_i) - W''(0)\Big)\Big].\end{aligned} \tag{6.8}$$

Then, with (6.7) and (6.8), the equation for v^ε at (t,x) can be written as

$$\begin{aligned}\mathrm{Eqn}(v^\varepsilon) &:= \varepsilon \partial_t v^\varepsilon(t,x) - \frac{1}{\varepsilon\,|\ln \varepsilon|}\left(\varepsilon \mathcal{I}_n[v^\varepsilon(t,\cdot)](x) - W'(v^\varepsilon(t,x))\right) \\ &= O(\varepsilon^{\frac{1}{2}}) + \sum_{i=1}^N \dot{\phi}_i \partial_t d_i(t,x) \\ &\quad - \frac{1}{\varepsilon\,|\ln\varepsilon|}\Bigg\{\sum_{i=1}^N \Big[(b_\varepsilon^i - a_\varepsilon^i) + W'(\tilde{\phi}_i)\Big] + O(\varepsilon) \\ &\quad + \varepsilon\,|\ln\varepsilon| \sum_{i=1}^N \Big[W''(\tilde{\phi}_i)\psi_i + \dot{\phi}_i c_0\left(\bar{a}_\varepsilon^i - \sigma\right) - \tilde{\sigma}\Big(W''(\tilde{\phi}_i) - W''(0)\Big)\Big] \\ &\quad - W'\left(\sum_{i=1}^N \tilde{\phi}_i + \varepsilon\,|\ln\varepsilon| \sum_{i=1}^N \psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\right)\Bigg\}.\end{aligned}$$

Grouping the error terms, the $\dot{\phi}_i$ terms, and the nonlinear terms together, we have

$$\begin{aligned}\mathrm{Eqn}(v^\varepsilon) &= O(|\ln\varepsilon|^{-1}) - \frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i=1}^N (b_\varepsilon^i - a_\varepsilon^i) + \sum_{i=1}^N \dot{\phi}_i \left[\partial_t d_i(t,x) - c_0\left(\bar{a}_\varepsilon^i - \sigma\right)\right] \\ &\quad + \frac{1}{\varepsilon\,|\ln\varepsilon|}\Bigg\{W'\left(\sum_{i=1}^N \tilde{\phi}_i + \varepsilon\,|\ln\varepsilon|\sum_{i=1}^N \psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\right) \\ &\qquad - \sum_{i=1}^N \Big(W'(\tilde{\phi}_i) + \varepsilon\,|\ln\varepsilon|\Big[W''(\tilde{\phi}_i)\psi_i - \tilde{\sigma}\Big(W''(\tilde{\phi}_i) - W''(0)\Big)\Big]\Big)\Bigg\}.\end{aligned} \tag{6.9}$$

Fix an index $i_0 \in \{1,\dots,N\}$. For the remainder of Step 1, we will conveniently isolate every term indexed with i_0 to help with Step 2. First, we do a Taylor expansion for W' around $\tilde{\phi}_{i_0}$ to obtain

$$\begin{aligned}& W'\Bigg(\sum_{i=1}^N \tilde{\phi}_i + \varepsilon\,|\ln\varepsilon|\sum_{i=1}^N \psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\Bigg) \\ &\qquad = W'(\tilde{\phi}_{i_0}) + W''(\tilde{\phi}_{i_0})\left(\sum_{i\neq i_0}\tilde{\phi}_i + \varepsilon\,|\ln\varepsilon|\sum_{i=1}^N \psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\right) \\ &\qquad\quad + O\left(\left(\sum_{i\neq i_0}\tilde{\phi}_i + \varepsilon\,|\ln\varepsilon|\sum_{i=1}^N \psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\right)^2\right).\end{aligned}$$

By (4.17) for ψ_i, we have that

$$\frac{1}{\varepsilon|\ln\varepsilon|}O\left(\left(\sum_{i\neq i_0}\tilde{\phi}_i+\varepsilon|\ln\varepsilon|\sum_{i=1}^N\psi_i-\tilde{\sigma}\varepsilon|\ln\varepsilon|\right)^2\right)$$
$$=\sum_{i\neq i_0}O\left(\frac{(\tilde{\phi}_i)^2}{\varepsilon|\ln\varepsilon|}\right)+O(|\ln\varepsilon|^{-1})+O(\varepsilon|\ln\varepsilon|).$$

Hence, we can write (6.9) as

$$\begin{aligned}\mathrm{Eqn}(v^\varepsilon)&=O(|\ln\varepsilon|^{-1})+\sum_{i\neq i_0}O\left(\frac{(\tilde{\phi}_i)^2}{\varepsilon|\ln\varepsilon|}\right)\\&\quad-\frac{1}{\varepsilon|\ln\varepsilon|}\sum_{i=1}^N(b_\varepsilon^i-a_\varepsilon^i)+\sum_{i=1}^N\dot{\phi}_i\left[\partial_t d_i(t,x)-c_0\left(\bar{a}_\varepsilon^i-\sigma\right)\right]\\&\quad+\frac{1}{\varepsilon|\ln\varepsilon|}\Bigg\{W'(\tilde{\phi}_{i_0})+W''(\tilde{\phi}_{i_0})\left(\sum_{i\neq i_0}\tilde{\phi}_i+\varepsilon|\ln\varepsilon|\sum_{i=1}^N\psi_i-\tilde{\sigma}\varepsilon|\ln\varepsilon|\right)\\&\quad-\left(W'(\tilde{\phi}_{i_0})+\varepsilon|\ln\varepsilon|\left[W''(\tilde{\phi}_{i_0})\psi_{i_0}-\tilde{\sigma}\left(W''(\tilde{\phi}_{i_0})-W''(0)\right)\right]\right)\\&\quad-\sum_{i\neq i_0}\left(W'(\tilde{\phi}_i)+\varepsilon|\ln\varepsilon|\left[W''(\tilde{\phi}_i)\psi_i-\tilde{\sigma}\left(W''(\tilde{\phi}_i)-W''(0)\right)\right]\right)\Bigg\}\end{aligned}$$

where in the last two lines we extracted the i_0 term. Cancelling the $W'(\tilde{\phi}_{i_0})$ and $W''(\tilde{\phi}_{i_0})\psi_{i_0}$ terms then distributing $1/(\varepsilon|\ln\varepsilon|)$, we simplify to

$$\begin{aligned}\mathrm{Eqn}(v^\varepsilon)&=O(|\ln\varepsilon|^{-1})+\sum_{i\neq i_0}O\left(\frac{(\tilde{\phi}_i)^2}{\varepsilon|\ln\varepsilon|}\right)\\&\quad-\frac{1}{\varepsilon|\ln\varepsilon|}\sum_{i=1}^N(b_\varepsilon^i-a_\varepsilon^i)+\sum_{i=1}^N\dot{\phi}_i\left[\partial_t d_i(t,x)-c_0\left(\bar{a}_\varepsilon^i-\sigma\right)\right]\\&\quad+W''(\tilde{\phi}_{i_0})\left(\sum_{i\neq i_0}\frac{\tilde{\phi}_i}{\varepsilon|\ln\varepsilon|}+\sum_{i\neq i_0}\psi_i-\tilde{\sigma}\right)+\tilde{\sigma}\left(W''(\tilde{\phi}_{i_0})-W''(0)\right)\\&\quad-\sum_{i\neq i_0}\left[\frac{W'(\tilde{\phi}_i)}{\varepsilon|\ln\varepsilon|}+W''(\tilde{\phi}_i)\psi_i-\tilde{\sigma}\left(W''(\tilde{\phi}_i)-W''(0)\right)\right].\end{aligned}$$

Next, we do a Taylor expansion for W' around 0 and recall that $W'(0)=0$ to write

$$W'(\tilde{\phi}_i)=W'(0)+W''(0)\tilde{\phi}_i+O((\tilde{\phi}_i)^2)=W''(0)\tilde{\phi}_i+O((\tilde{\phi}_i)^2).$$

With this, we now have that

$$\begin{aligned}\mathrm{Eqn}(v^\varepsilon)&=O(|\ln\varepsilon|^{-1})+\sum_{i\neq i_0}O\left(\frac{(\tilde{\phi}_i)^2}{\varepsilon|\ln\varepsilon|}\right)\\&\quad-\frac{1}{\varepsilon|\ln\varepsilon|}\sum_{i=1}^N(b_\varepsilon^i-a_\varepsilon^i)+\sum_{i=1}^N\dot{\phi}_i\left[\partial_t d_i(t,x)-c_0\left(\bar{a}_\varepsilon^i-\sigma\right)\right]\end{aligned}$$

$$+ W''(\tilde{\phi}_{i_0})\left(\sum_{i\neq i_0}\frac{\tilde{\phi}_i}{\varepsilon\,|\ln\varepsilon|}+\sum_{i\neq i_0}\psi_i-\tilde{\sigma}\right)+\tilde{\sigma}\left(W''(\tilde{\phi}_{i_0})-W''(0)\right)$$
$$-\sum_{i\neq i_0}\left[\frac{W''(0)\tilde{\phi}_i}{\varepsilon\,|\ln\varepsilon|}+W''(\tilde{\phi}_i)\psi_i-\tilde{\sigma}\left(W''(\tilde{\phi}_i)-W''(0)\right)\right].$$

We rearrange to group the terms with $W''(\tilde{\phi}_{i_0})-W''(0)$ together

$$\begin{aligned}\mathrm{Eqn}(v^\varepsilon) &= O(|\ln\varepsilon|^{-1})+\sum_{i\neq i_0}O\left(\frac{(\tilde{\phi}_i)^2}{\varepsilon\,|\ln\varepsilon|}\right)\\ &\quad-\frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i=1}^N(b^i_\varepsilon-a^i_\varepsilon)+\sum_{i=1}^N\dot{\phi}_i\left[\partial_t d_i(t,x)-c_0\left(\bar{a}^i_\varepsilon-\sigma\right)\right]\\ &\quad+\left(W''(\tilde{\phi}_{i_0})-W''(0)\right)\sum_{i\neq i_0}\frac{\tilde{\phi}_i}{\varepsilon\,|\ln\varepsilon|}-\tilde{\sigma}W''(0)\\ &\quad+\sum_{i\neq i_0}\left[\left(W''(\tilde{\phi}_{i_0})-W''(\tilde{\phi}_i)\right)\psi_i+\tilde{\sigma}\left(W''(\tilde{\phi}_i)-W''(0)\right)\right].\end{aligned}\tag{6.10}$$

Looking at the last line, we Taylor expand W'' around 0 to find, for $i\neq i_0$,

$$\begin{aligned}(W''(\tilde{\phi}_{i_0})-W''(\tilde{\phi}_i))\psi_i+\tilde{\sigma}\left(W''(\tilde{\phi}_i)-W''(0)\right) &= O(\psi_i)-\tilde{\sigma}\left(W'''(0)\tilde{\phi}_i+O(\tilde{\phi}_i^2)\right)\\ &= O(\psi_i)+O(\tilde{\phi}_i)\end{aligned}$$

and also

$$W''(\tilde{\phi}_{i_0})-W''(0)=W'''(0)\tilde{\phi}_{i_0}+O((\tilde{\phi}_{i_0})^2)=O(\tilde{\phi}_{i_0}).$$

Therefore, (6.10) can be written as

$$\begin{aligned}\mathrm{Eqn}(v^\varepsilon) &= O(|\ln\varepsilon|^{-1})+\sum_{i\neq i_0}\left[O\left(\frac{(\tilde{\phi}_i)^2}{\varepsilon\,|\ln\varepsilon|}\right)+O(\tilde{\phi}_i)+O(\psi_i)\right]\\ &\quad-\frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i=1}^N(b^i_\varepsilon-a^i_\varepsilon)+\sum_{i=1}^N\dot{\phi}_i\left[\partial_t d_i(t,x)-c_0\left(\bar{a}^i_\varepsilon-\sigma\right)\right]+O(\tilde{\phi}_{i_0})\sum_{i\neq i_0}\frac{\tilde{\phi}_i}{\varepsilon\,|\ln\varepsilon|}-\sigma\end{aligned}$$

where we also used that $\sigma=W''(0)\tilde{\sigma}$. Hence, we conclude that

$$\begin{aligned}\mathrm{Eqn}(v^\varepsilon) &= O(|\ln\varepsilon|^{-1})\\ &\quad+\sum_{i\neq i_0}\left[O\left(\frac{(\tilde{\phi}_i)^2}{\varepsilon\,|\ln\varepsilon|}\right)+O(\tilde{\phi}_i)+O(\dot{\phi}_i)+O(\psi_i)+\frac{O(\tilde{\phi}_{i_0})\tilde{\phi}_i}{\varepsilon\,|\ln\varepsilon|}\right]\\ &\quad-\frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i=1}^N(b^i_\varepsilon-a^i_\varepsilon)+\sum_{i=1}^N\dot{\phi}_i\left[\partial_t d_i(t,x)-c_0\left(\bar{a}^i_\varepsilon-\sigma\right)\right]-\sigma.\end{aligned}\tag{6.11}$$

Step 2. $v^\varepsilon(t,x)$ satisfies (6.4) when (t,x) is near the front $\Gamma^{i_0}_t$.

Assume that $|d_{i_0}(t,x)-\sigma|\le|\ln\varepsilon|^{-1/2}$ for some index $1\le i_0\le N$. Then, by (6.1), for ε sufficiently small,

$$|d_i(t,x)-\sigma|\ge|\ln\varepsilon|^{-\frac{1}{2}}\quad\text{for all } i\neq i_0.$$

We begin by estimating the error terms in (6.11) for $i \neq i_0$. First, we use (4.2) to estimate

$$|\tilde{\phi}_i| = \left| \phi\left(\frac{d_i(t,x)-\sigma}{\varepsilon}\right) - H\left(\frac{d_i(t,x)-\sigma}{\varepsilon}\right)\right| \leq C\frac{\varepsilon}{|d_i(t,x)-\sigma|} = O(\varepsilon\,|\ln\varepsilon|^{\frac{1}{2}}),$$

from which it follows that

$$\frac{|\tilde{\phi}_i|}{\varepsilon\,|\ln\varepsilon|} = O\left(|\ln\varepsilon|^{-\frac{1}{2}}\right) \quad \text{and} \quad \frac{|\tilde{\phi}_i|^2}{\varepsilon\,|\ln\varepsilon|} = O(\varepsilon).$$

Similarly, from (4.18), for $i \neq i_0$,

$$|\psi_i| \leq \frac{C}{\varepsilon|\ln\varepsilon|}\frac{\varepsilon}{|d_i(t,x)-\sigma|} = O\left(|\ln\varepsilon|^{-\frac{1}{2}}\right).$$

Next, we use (4.3) to find that, for $i \neq i_0$,

$$|\dot{\phi}_i| \leq \frac{C\varepsilon^2}{|d_i(t,x)-\sigma|^2} = O(\varepsilon^2\,|\ln\varepsilon|).$$

Combining the above estimates in view of (6.11), we have

$$\sum_{i\neq i_0}\left(O\left(\frac{(\tilde{\phi}_i)^2}{\varepsilon\,|\ln\varepsilon|}\right) + O(\tilde{\phi}_i) + O(\dot{\phi}_i) + O(\psi_i) + \frac{O(\tilde{\phi}_{i_0})\tilde{\phi}_i}{\varepsilon\,|\ln\varepsilon|}\right) = O\left(|\ln\varepsilon|^{-\frac{1}{2}}\right).$$

Next, we check the terms with $\bar{a}^i_\varepsilon$ and a^i_ε. For $i \neq i_0$, we use that d_i is smooth, (4.3) and (4.10) to obtain

$$\left|\sum_{i\neq i_0}\dot{\phi}_i\left[\partial_t d_i(t,x) - c_0\left(\bar{a}^i_\varepsilon - \sigma\right)\right]\right| \leq \sum_{i\neq i_0}\left(\frac{\varepsilon}{d_i(t,x)-\sigma}\right)^2\frac{C}{\varepsilon^{\frac{1}{2}}|\ln\varepsilon|} = O\left(\varepsilon^{\frac{3}{2}}\right). \tag{6.12}$$

For $i = i_0$, we use that $\dot{\phi}_{i_0} \geq 0$, (6.2), and Lemma 4.4 to estimate

$$\begin{aligned}\dot{\phi}_{i_0}[\partial_t d_{i_0}(t,x) - c_0(\bar{a}^{i_0}_\varepsilon - \sigma)] &= \dot{\phi}_{i_0}\left([\partial_t d_{i_0}(t,x) - \mu\Delta d_{i_0}(t,x) + c_0\sigma] + [\mu\Delta d_{i_0}(t,x) - c_0\bar{a}^{i_0}_\varepsilon]\right)\\ &\leq \dot{\phi}_{i_0}(0 + o_\varepsilon(1)) = o_\varepsilon(1).\end{aligned}$$

Lastly, by Lemma 4.5 and Lemma 4.6

$$\frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i=1}^N\left|b^i_\varepsilon - a^i_\varepsilon\right| = \frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i\neq i_0}\left|b^i_\varepsilon - a^i_\varepsilon\right| \leq \frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i\neq i_0}\frac{C\varepsilon}{|\ln\varepsilon|^{-\frac{1}{2}}} = O(|\ln\varepsilon|^{-\frac{1}{2}}). \tag{6.13}$$

Consequently, in (6.11), we have that

$$\mathrm{Eqn}(v^\varepsilon) \leq o_\varepsilon(1) - \sigma.$$

Taking ε sufficiently small, (6.4) holds.

<u>**Step 3**</u>. $v^\varepsilon(t,x)$ satisfies (6.4) when (t,x) is away from all fronts Γ^i_t.

Assume that for all $i = 1,\dots,N$,

$$|d_i(t,x) - \sigma| \geq |\ln\varepsilon|^{-\frac{1}{2}}.$$

Then, we estimate exactly as in Step 2 but we include $i = i_0$ in (6.12) and do not drop $i = i_0$ in (6.13). Consequently, we have that

$$\mathrm{Eqn}(v^\varepsilon) \leq o_\varepsilon(1) - \sigma.$$

Taking ε sufficiently small, (6.4) holds.

<u>**Step 4**</u>. $v^\varepsilon(t,x)$ satisfies (6.5).

Let $\tilde{\rho} := \frac{C}{\tilde{\sigma}|\ln\varepsilon|}$ for $C > 0$ to be determined and $\varepsilon > 0$ sufficiently small. Fix (t, x) such that $d_N(t,x) - \tilde{\sigma} \geq \tilde{\rho}$. Then,

$$d_i(t,x) - \tilde{\sigma} \geq \tilde{\rho} \quad \text{for all } 1 \leq i \leq N. \tag{6.14}$$

Since $0 < \phi < 1$ and by (4.18),

$$\begin{aligned} v^\varepsilon(t,x) &\leq N + C\sum_{i=1}^N \frac{\varepsilon}{|d_i(t,x) - \tilde{\sigma}|} - \tilde{\sigma}\varepsilon|\ln\varepsilon| \\ &\leq N + \frac{C\varepsilon}{\tilde{\rho}} - \tilde{\sigma}\varepsilon|\ln\varepsilon| \\ &\leq N - \frac{\tilde{\sigma}}{2}\varepsilon|\ln\varepsilon| \end{aligned}$$

for a sufficiently large C in the definition of $\tilde{\rho}$. On the other hand, by (6.14), (4.2) and (4.18),

$$\begin{aligned} v^\varepsilon(t,x) &\geq N + \sum_{i=1}^N \left(\phi\left(\frac{d_i(t,x) - \tilde{\sigma}}{\varepsilon}\right) - H\left(\frac{d_i(t,x) - \tilde{\sigma}}{\varepsilon}\right)\right) - C\sum_{i=1}^N \frac{\varepsilon}{|d_i(t,x) - \tilde{\sigma}|} - \tilde{\sigma}\varepsilon|\ln\varepsilon| \\ &\geq N - \sum_{i=1}^N \left(\frac{\varepsilon}{\alpha(d_i(t,x) - \tilde{\sigma})} + \frac{C\varepsilon^2}{(d_i(t,x) - \tilde{\sigma})^2}\right) - C\sum_{i=1}^N \frac{\varepsilon}{|d_i(t,x) - \tilde{\sigma}|} - \tilde{\sigma}\varepsilon|\ln\varepsilon| \\ &\geq N - \frac{C\varepsilon}{\rho} - \frac{C\varepsilon^2}{\rho^2} - \tilde{\sigma}\varepsilon|\ln\varepsilon| \\ &\geq N - 2\tilde{\sigma}\varepsilon|\ln\varepsilon| \end{aligned}$$

for C sufficiently large in the choice of $\tilde{\rho}$. This proves (6.5). □

7. Proof of Theorem 1.1

Proof. We apply an adaptation of the abstract method introduced in [3], see also [1].

Begin by defining the families of open sets $(D^i)_{i=1}^N$ and $(E^i)_{i=1}^N$ by

$$D^i = \operatorname{Int}\left\{(t,x) \in (0,\infty) \times \mathbb{R}^n : \liminf_{\varepsilon\to 0}{}_* \frac{u^\varepsilon - i}{\varepsilon\,|\ln\varepsilon|} \geq 0\right\} \subset (0,\infty) \times \mathbb{R}^n$$

$$E^i = \operatorname{Int}\left\{(t,x) \in (0,\infty) \times \mathbb{R}^n : \limsup_{\varepsilon\to 0}{}^* \frac{u^\varepsilon - (i-1)}{\varepsilon\,|\ln\varepsilon|} \leq 0\right\} \subset (0,\infty) \times \mathbb{R}^n.$$

To define the traces of D^i and E^i, we first define the functions $\underline{\chi}^i, \overline{\chi}^i : (0,\infty) \times \mathbb{R}^n \to \{-1, 1\}$, respectively, by

$$\underline{\chi}^i = \mathbb{1}_{D^i} - \mathbb{1}_{(D^i)^c} \quad \text{and} \quad \overline{\chi}^i = \mathbb{1}_{(E^i)^c} - \mathbb{1}_{E^i}.$$

Since D^i is open, $\underline{\chi}^i$ is lower semicontinuous, and since $(E_i)^c$ is closed, $\overline{\chi}^i$ is upper semicontinuous. To ensure that $\overline{\chi}^i$ and $\underline{\chi}^i$ remain lower and upper semicontinuous, respectively, at $t = 0$, we set

$$\underline{\chi}^i(0,x) = \liminf_{t\to 0,\ y\to x} \underline{\chi}^i(t,y) \quad \text{and} \quad \overline{\chi}^i(0,x) = \limsup_{t\to 0,\ y\to x} \overline{\chi}^i(t,y).$$

Define the traces D_0^i and E_0^i by

$$D_0^i = \{x \in \mathbb{R}^n : \underline{\chi}^i(0,x) = 1\} \quad \text{and} \quad E_0^i = \{x \in \mathbb{R}^n : \overline{\chi}^i(0,x) = -1\}.$$

Note that D_0^i and E_0^i are open sets. To apply the abstract method, we need the following propositions. We delay their proofs until the end of the section.

Proposition 7.1 (Initialization). *For each $i = 1, \dots, N$,*

$$\Omega_0^i \subset D_0^i \quad \textit{and} \quad (\overline{\Omega}_0^i)^c \subset E_0^i.$$

Proposition 7.2 (Propagation). *For each $i = 1, \dots, N$, the set D^i is a generalized super-flow, and the set $\overline{E^i}$ is a generalized sub-flow.*

For $t > 0$, define the sets D_t^i and E_t^i by

$$D_t^i = \{x \in \mathbb{R}^n : (t, x) \in D^i\} \quad \text{and} \quad E_t^i = \{x \in \mathbb{R}^n : (t, x) \in E^i\}.$$

By the abstract method (see [1, 3]), it follows from Propositions 7.1 and 7.2 that

$${}^+\Omega_t^i \subset D_t^i \subset {}^+\Omega_t^i \cup \Gamma_t^i \quad \text{and} \quad {}^-\Omega_t^i \subset E_t^i \subset {}^-\Omega_t^i \cup \Gamma_t^i.$$

The conclusion readily follows; we provide the details for completeness.

First, since ${}^+\Omega_t^i \subset D_t^i$, we use the definition of D_t^i to see that

$$\liminf_{\varepsilon \to 0} {}_*u^\varepsilon(t, x) \geq i \quad \text{for } x \in {}^+\Omega_t^i. \tag{7.1}$$

Using that ${}^-\Omega_t^{i+1} \subset E_t^{i+1}$, we similarly get

$$\limsup_{\varepsilon \to 0} {}^*u^\varepsilon(t, x) \leq (i + 1) - 1 = i \quad \text{for } x \in {}^-\Omega_t^{i+1}. \tag{7.2}$$

Therefore, for $i = 1, \dots, N - 1$,

$$\lim_{\varepsilon \to 0} u^\varepsilon(t, x) = i \quad \text{in } {}^+\Omega_t^i \cap {}^-\Omega_t^{i+1}.$$

Next, by the comparison principle, $0 \leq u^\varepsilon \leq N$. Consequently,

$$0 \leq \liminf_{\varepsilon \to 0} {}_*u^\varepsilon \quad \text{and} \quad \limsup_{\varepsilon \to 0} {}^*u^\varepsilon \leq N.$$

Hence, together with (7.2) and respectively (7.1) we have

$$\lim_{\varepsilon \to 0} u^\varepsilon(t, x) = 0 \quad \text{in } {}^-\Omega_t^1 \quad \text{and} \quad \lim_{\varepsilon \to 0} u^\varepsilon(t, x) = N \quad \text{in } {}^+\Omega_t^N.$$

□

It remains to prove Propositions 7.1 and 7.2. We begin with the initialization.

7.1. Proof of Proposition 7.1.

Proof. We will prove that $\Omega_0^{i_0} \subset D_0^{i_0}$ for all $1 \leq i_0 \leq N$. The proof of $(\overline{\Omega}_0^{i_0})^c \subset E_0^{i_0}$ is similar.

Fix i_0, a point $x_0 \in \Omega_0^{i_0}$, and a small constant $\tilde{\sigma} > 0$. To prove that $x_0 \in D_0^{i_0}$, it is enough to show that, for all (t, x) in a neighborhood of $(0, x_0)$ in $[0, \infty) \times \mathbb{R}^n$,

$$\liminf_{\varepsilon \to 0} {}_*\frac{u^\varepsilon(t, x) - i_0}{\varepsilon\,|\ln \varepsilon|} \geq 0.$$

For this, we will use (6.3) to construct a suitable subsolution $v^\varepsilon \leq u^\varepsilon$, depending on $\tilde{\sigma}$.

We begin by defining smooth functions φ_i for each $i = 1, \dots, i_0$ that satisfy conditions (i),(ii),(iii) in Definition 2.1. Indeed, first let $r_i > 0$ be given by

$$r_i = d_i(x_0) - \frac{\tilde{\sigma}}{2}, \quad i = 1, \dots, i_0,$$

where d_i is given in (1.5). Note that $B(x_0, r_i) \subset\subset \Omega_0^i$ and

$$r_i - r_{i+1} = d_i(x_0) - d_{i+1}(x_0) \geq d(\Gamma_0^i, \Gamma_0^{i+1}).$$

Define the smooth functions $\varphi_i(x)$, $i = 1, \dots, i_0$, by

$$\varphi_i(t, x) = (r_i - Ct)_+^2 - |x - x_0|^2$$

for a large constant $C > 0$, to be determined. It is easy to check that the signed distance function $\tilde{d}_i(t,x)$ associated to $\{x : \varphi_i(t,x) > 0\}$ is

$$\tilde{d}_i(t,x) = r_i - Ct - |x - x_0| \tag{7.3}$$

and that

$$\{(t,x) : \varphi_i(t,x) > 0\} = \bigcup_{t\geq 0}\{t\} \times B(x_0, r_i - Ct).$$

Hence, (i) in Definition 2.1 is satisfied. Next, we see that

$$\partial_t\varphi_i(t,x) = -2C(r_i - Ct), \quad \nabla t\varphi_i(t,x) = -2(x - x_0)$$

and, for $t < r_{i_0}/(2C)$, we have

$$\begin{aligned}\partial_t\varphi_i - \mu\,\mathrm{tr}\left((I - \widehat{\nabla\varphi_i} \otimes \widehat{\nabla\varphi_i})D^2\varphi_i\right) &= -2C(r_i - Ct) + 2\mu\,(n-1)\\ &\leq -Cr_{i_0} + 2\mu\,(n-1)\\ &\leq -c_0\sigma\end{aligned}$$

for $C > 0$ sufficiently large, with $\sigma = W''(0)\tilde{\sigma}$. Hence, $(ii), (iii)$ in Definition 2.1 are also satisfied. Moreover, by construction,

$$\tilde{d}(0,x) \leq d(x) - \frac{\tilde{\sigma}}{2} \quad \text{for all } x \in \mathbb{R}^n.$$

Let $\rho < \tilde{\sigma}/2$, to be determined, and let $\bar{d}_i$ be the smooth, bounded extension of $\tilde{d}_i$ outside of Q^i_ρ in Definition 4.3. Since $\bar{d}_i \leq \tilde{d}_i + \rho$, we have that

$$\bar{d}(0,x) \leq d(x) \quad \text{for all } x \in \mathbb{R}^n. \tag{7.4}$$

Let $v^\varepsilon = v^\varepsilon(t,x)$ be given by

$$v^\varepsilon(t,x) = \sum_{i=1}^{i_0}\phi\left(\frac{\bar{d}_i(t,x) - \tilde{\sigma}}{\varepsilon}\right) + \varepsilon\,|\ln\varepsilon|\sum_{i=1}^{i_0}\psi_i\left(\frac{\bar{d}_i(t,x) - \tilde{\sigma}}{\varepsilon}; t, x\right) - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\,.$$

Note that v^ε is the same as (6.3) except we only sum over $1 \leq i \leq i_0$, i.e. $N = i_0$ in Lemma 6.1. By Lemma 6.1, we have that v^ε is a subsolution to (6.4) in $[0, r_{i_0}/(2C)] \times \mathbb{R}^n$.

We claim that $v^\varepsilon \leq u^\varepsilon$ in a neighborhood $\mathcal{N}(0,x_0) \subset \{\bar{d}_{i_0}(t,x) - \tilde{\sigma} \geq \rho\}$ where $\rho = 2C/(\tilde{\sigma}|\ln\varepsilon|)$. Note that $\rho < \tilde{\sigma}$ for ε sufficiently small. Let x be such that $\bar{d}_{i_0}(0,x) - \tilde{\sigma} \geq \rho$. Then

$$d_i(x) \geq d_{i_0}(x) \geq d_{i_0}(x_0) - |x - x_0| = \bar{d}_{i_0}(0,x) + \frac{\tilde{\sigma}}{2} \geq \frac{3\tilde{\sigma}}{2} + \rho \geq \tilde{\sigma}$$

for all $1 \leq i \leq i_0$. Since $\phi \geq 0$ and by (4.2),

$$\begin{aligned}u^\varepsilon(0,x) &\geq \sum_{i=1}^{i_0}\phi\left(\frac{d_i(x)}{\varepsilon}\right)\\ &\geq \sum_{i=1}^{i_0}\left(1 - \frac{\varepsilon}{\alpha d_i(x)} - \frac{C\varepsilon^2}{(d_i(x))^2}\right)\\ &\geq \sum_{i=1}^{i_0}\left(1 - \frac{\varepsilon}{\alpha\tilde{\sigma}} - \frac{C\varepsilon^2}{\tilde{\sigma}^2}\right)\\ &\geq i_0 - \left(\frac{2\varepsilon}{\alpha\tilde{\sigma}} + \frac{4C\varepsilon^2}{\tilde{\sigma}^2}\right)N\end{aligned}$$

$$\geq i_0 - \frac{\tilde{\sigma}}{2}\varepsilon\,|\ln\varepsilon|\,,$$

for ε sufficiently small. Therefore, by the second inequality in (6.5) (applied with $N = i_0$),

$$u^\varepsilon(0,x) \geq i_0 - \frac{\tilde{\sigma}}{2}\varepsilon\,|\ln\varepsilon| \geq v^\varepsilon(0,x).$$

By the comparison principle, the claim holds. Consequently, by the first inequality in (6.5),

$$\liminf_{\varepsilon\to 0}{}_*\frac{u^\varepsilon(t,x)-i_0}{\varepsilon\,|\ln\varepsilon|} \geq \liminf_{\varepsilon\to 0}{}_*\frac{v^\varepsilon(t,x)-i_0}{\varepsilon\,|\ln\varepsilon|} \geq -2\tilde{\sigma} \quad \text{in } \mathcal{N}(0,x_0)\subset\{\bar{d}_{i_0}(t,x)-\tilde{\sigma}\geq 0\}.$$

Letting $\tilde{\sigma}\to 0$, the result follows. □

7.2. **Proof of Proposition 7.2.**

Proof. Fix $1 \leq i_0 \leq N$. We will show that D^{i_0} is a generalized super-flow. The proof that $\overline{E^{i_0}}$ is a generalized sub-flow is similar.

Let $(t_0,x_0)\in(0,\infty)\times\mathbb{R}^n$, $h>0$, and $\varphi_{i_0}:[t_0,t_0+h]\times\mathbb{R}^n\to\mathbb{R}$ be a smooth function satisfying (i)-(iv) in Definition 2.1. If $i_0\neq 1$, we will construct a smooth test function φ_i for the generalized flows D^i, $1\leq i<i_0$. If $i_0=1$, then we omit this step.

Consider the sequence of smooth functions φ_1^k given by

$$\varphi_1^k(t,x) = \varphi_{i_0}(t,x) + \frac{1}{k}.$$

Since $\nabla\varphi_{i_0}\neq 0$ on $\{\varphi_{i_0}=0\}$ and φ_{i_0} is smooth,

$$\nabla\varphi_1^k = \nabla\varphi_{i_0}\neq 0 \quad \text{in } \left\{-\frac{1}{k}\leq\varphi_{i_0}\leq\frac{1}{k}\right\}$$

for sufficiently large k. Consequently,

$$\partial\{\varphi_1^k\geq 0\} = \{\varphi_1^k = 0\}$$

for large enough k. Then, the sequence of sets $(\{\varphi_1^k\geq 0\})_{k\in\mathbb{N}}$ is strictly decreasing and

$$\lim_{k\to\infty}\{\varphi_1^k\geq 0\} = \bigcap_{k\geq 1}\{\varphi_1^k\geq 0\} = \{\varphi_{i_0}\geq 0\}.$$

Recall from (i) that $\{\varphi_{i_0}\geq 0\}\subset[t_0,t_0+h]\times B(x_0,r)$. Since $\{\varphi_{i_0}\geq 0\}$ is closed and $B(x_0,r)$ is open, for k sufficiently large, we have

$$\{\varphi_{i_0}\geq 0\}\subset\{\varphi_1^k\geq 0\}\subset[t_0,t_0+h]\times B(x_0,r).$$

Moreover, since the mean curvature equation is geometric and φ_{i_0} satisfies (iii), we have

$$\partial_t\varphi_1^k + F^*(\nabla\varphi_1^k, D^2\varphi_1^k) = \partial_t\varphi_{i_0} + F^*(\nabla\varphi_{i_0}, D^2\varphi_{i_0}) \leq -\delta$$

in $[t_0,t_0+h]\times\overline{B}(x_0,r)$. Therefore, φ_1^k satisfies $(i),(ii),(iii)$ in Definition 2.1 when k is large.

For $i=1,\dots,i_0-1$, we define the smooth functions $\varphi_i:[t_0,t_0+h]\times\mathbb{R}^n\to\mathbb{R}$ by

$$\varphi_i(t,x) := \varphi_{i_0}(t,x) + \frac{1}{k}\left(\frac{i_0-i}{i_0-1}\right),$$

for a sufficiently large k. As a consequence of our previous discussion, each φ_i satisfies (i),(ii),(iii) in Definition 2.1. Moreover, (6.1) holds and, since $D^i_{t_0}$ is open,

$$\{x\in\overline{B}(x,r):\varphi_i(t_0,x)\geq 0\} = \left\{x\in\overline{B}(x,r):\varphi_{i_0}(t_0,x)\geq -\frac{1}{k}\left(\frac{i_0-i}{i_0-1}\right)\right\}\subset D^i_{t_0}$$

by making k larger, if necessary. Therefore, each φ_i also satisfies (iv) in Definition 2.1.

Let $\tilde{d}_i(t,x)$ be the signed distance function associated to $\{(t,x) : \varphi_i(t,x) \geq 0\}$. Let $\rho > 0$ and d_i be such that d_i is a smooth, bounded extension of $\tilde{d}_i$ outside of Q^i_ρ as in Definition 4.3. Let $v^\varepsilon = v^\varepsilon(t,x)$ be given by

$$v^\varepsilon(t,x) = \sum_{i=1}^{i_0} \phi\left(\frac{d_i(t,x) - \tilde{\sigma}}{\varepsilon}\right) + \varepsilon\left|\ln\varepsilon\right| \sum_{i=1}^{i_0} \psi_i\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}; t, x\right) - \tilde{\sigma}\varepsilon\left|\ln\varepsilon\right|.$$

In light of (6.3), we note that $N = i_0$ in Lemma 6.1. In particular, we apply Lemma 6.1 to conclude that v^ε is a subsolution to (6.4) in $[t_0, t_0+h] \times \mathbb{R}^n$.

We next show that

$$u^\varepsilon \geq v^\varepsilon \quad \text{in } \{(t,x) \in [t_0, t_0+h] \times \mathbb{R}^n : d_{i_0}(t,x) - \tilde{\sigma} \geq \rho\} \tag{7.5}$$

where $2C/(\tilde{\sigma}|\ln\varepsilon|) \leq \rho \leq \tilde{\sigma}$ for ε sufficiently small. By the initial condition (iv) in Definition 2.1, we have that

$$\{x : d_{i_0}(t_0,x) \geq 0\} = \{x : \varphi_{i_0}(t_0,x) \geq 0\} \subset D^{i_0}_{t_0} = \left\{x : \liminf_{\varepsilon\to 0}{}_* \frac{u^\varepsilon(t_0,x) - i_0}{\varepsilon\left|\ln\varepsilon\right|} \geq 0\right\}.$$

Therefore,

$$\{x : d_{i_0}(t_0,x) - \tilde{\sigma} \geq \rho\} \subset \left\{x : \liminf_{\varepsilon\to 0}{}_* \frac{u^\varepsilon(t_0,x) - i_0}{\varepsilon\left|\ln\varepsilon\right|} \geq 0\right\},$$

which further gives that

$$u^\varepsilon(t_0,x) \geq i_0 - \frac{\tilde{\sigma}}{2}\varepsilon\left|\ln\varepsilon\right| \quad \text{in } \{x : d_{i_0}(t_0,x) - \tilde{\sigma} \geq \rho\}.$$

In particular, by the second inequality in (6.5),

$$u^\varepsilon(t_0,x) \geq i_0 - \frac{\tilde{\sigma}}{2}\varepsilon\left|\ln\varepsilon\right| \geq v^\varepsilon(t_0,x).$$

By the comparison principle, (7.5) holds.

By the first inequality in (6.5), we have that

$$\frac{u^\varepsilon(t_0+h,x) - i_0}{\varepsilon\left|\ln\varepsilon\right|} \geq \frac{v^\varepsilon(t_0+h,x) - i_0}{\varepsilon\left|\ln\varepsilon\right|} \geq -2\tilde{\sigma} \quad \text{in } \{x : d_{i_0}(t_0+h,x) - \tilde{\sigma} \geq \rho\}.$$

Since $\rho \to 0$ as $\varepsilon \to 0$, it follows that

$$\{x : d_{i_0}(t_0+h,x) - \tilde{\sigma} \geq 0\} \subset \left\{x : \liminf_{\varepsilon\to 0}{}_* \frac{u^\varepsilon(t_0+h,x) - i_0}{\varepsilon\left|\ln\varepsilon\right|} \geq -2\tilde{\sigma}\right\},$$

Taking $\tilde{\sigma} \to 0$, we arrive at the desired inclusion

$$\{x : \varphi_{i_0}(t_0+h,x) \geq 0\} = \{x : d_{i_0}(t_0+h,x) \geq 0\} \subset \left\{x : \liminf_{\varepsilon\to 0}{}_* \frac{u^\varepsilon(t_0+h,x) - i_0}{\varepsilon\left|\ln\varepsilon\right|} \geq 0\right\}.$$

□

8. Proof of Lemma 4.2

First, we note that by the regularity of ϕ and d, there exists $C_0 > 0$ such that

$$\begin{aligned} &\left|\phi\left(\xi + \frac{d(t,x+\varepsilon z) - d(t,x)}{\varepsilon}\right) - \phi\left(\xi + \nabla d(t,x)\cdot z\right)\right| \\ &\leq C_0 \dot{\phi}(\xi + \nabla d(t,x)\cdot z + \varepsilon O(|z|^2))\varepsilon|z|^2, \qquad \text{with } |O(|z|^2)| \leq C_0|z|^2. \end{aligned} \tag{8.1}$$

8.1. **Proof of** (4.6) **and** (4.10)**.** To prove (4.6), we will show that for all $(t,x)\in(0,\infty)\times\mathbb{R}^n$,

$$|a_\varepsilon(\xi;t,x)|\le C\varepsilon^{\frac12}. \tag{8.2}$$

The same estimate for $\dot a_\varepsilon$, $\ddot a_\varepsilon$, $\dddot a_\varepsilon$, and $a_\varepsilon^{(iv)}$ will follow similarly by replacing ϕ in expression (4.4) by, respectively, $\dot\phi$, $\ddot\phi$, $\dddot\phi$ and $\phi^{(iv)}$.

Begin by writing

$$\begin{aligned} a_\varepsilon &= \int_{|z|<\varepsilon^{-\frac12}}\left(\phi\left(\xi+\frac{d(t,x+\varepsilon z)-d(t,x)}{\varepsilon}\right)-\phi\left(\xi+\nabla d(t,x)\cdot z\right)\right)\frac{dz}{|z|^{n+1}}\\ &\quad+\int_{|z|>\varepsilon^{-\frac12}}\left(\phi\left(\xi+\frac{d(t,x+\varepsilon z)-d(t,x)}{\varepsilon}\right)-\phi\left(\xi+\nabla d(t,x)\cdot z\right)\right)\frac{dz}{|z|^{n+1}}\\ &=: I+II. \end{aligned}$$

For the long-range interactions,

$$|II|\le 2\int_{|z|>\varepsilon^{-\frac12}}\frac{dz}{|z|^{n+1}}=C\varepsilon^{\frac12}.$$

For the short range interactions, we use (8.1) to estimate

$$|I|\le C\int_{|z|<\varepsilon^{-\frac12}}\varepsilon\,|z|^2\,\frac{dz}{|z|^{n+1}}=C\varepsilon^{\frac12}.$$

Estimate (8.2) follows.

Consequently, using the behavior of ϕ at $\pm\infty$ in (1.4), we estimate

$$|\bar a_\varepsilon(t,x)|\le\frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}}|a_\varepsilon(\xi;t,x)|\,\dot\phi(\xi)\,d\xi\le\frac{C\varepsilon^{\frac12}}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}}\dot\phi(\xi)\,d\xi=\frac{C}{\varepsilon^{\frac12}\,|\ln\varepsilon|}, \tag{8.3}$$

which gives (4.10).

8.2. **Proof of** (4.8)**.** Let C_0 be as in (8.1). We will consider two cases.

<u>*Case 1*</u>: $|\xi|\le\frac{4\|d\|_\infty}{\varepsilon}$. Choose $\kappa>0$ such that

$$\|\nabla d\|_\infty\kappa<\frac14\quad\text{and}\quad 4\|d\|_\infty C_0\kappa^2\le\frac14.$$

Then, for $|z|<\kappa|\xi|$, we have that $|\nabla d(t,x)\cdot z|\le|\xi|/4$ and $C_0\varepsilon|z|^2\le|\xi|/4$. In particular,

$$|\xi+\nabla d(t,x)\cdot z+C_0\varepsilon|z|^2|\ge\frac{|\xi|}{2}. \tag{8.4}$$

We write

$$\begin{aligned} a_\varepsilon &= \int_{\mathbb{R}^n}\left(\phi\left(\xi+\frac{d(t,x+\varepsilon z)-d(t,x)}{\varepsilon}\right)-\phi\left(\xi+\nabla d(t,x)\cdot z\right)\right)\frac{dz}{|z|^{n+1}}\\ &=\int_{\{|z|>\kappa|\xi|\}}(\dots)+\int_{\{|z|<\kappa|\xi|\}}(\dots)\\ &=: I+II. \end{aligned}$$

By (8.1), we have

$$|I|\le\int_{\{|z|<\kappa|\xi|\}}C_0\dot\phi(\xi+\nabla d(t,x)\cdot z+\varepsilon O(|z|^2))\varepsilon|z|^2\frac{dz}{|z|^{n+1}},$$

and by the asymptotic estimates (4.3) and (8.4),

$$\dot{\phi}(\xi + \nabla d(t,x)\cdot z + \varepsilon O(|z|^2)) \leq \frac{C}{(\xi + \nabla d(t,x)\cdot z + \varepsilon O(|z|^2))^2} \leq \frac{C}{|\xi|^2}.$$

Therefore,

$$|I| \leq \frac{C\varepsilon}{|\xi|^2}\int_{\{|z|<\kappa|\xi|\}} \frac{dz}{|z|^{n-1}} = \frac{C\varepsilon}{|\xi|}.$$

Next, we have

$$|II| \leq 2\int_{\{|z|>\kappa|\xi|\}} \frac{dz}{|z|^{n+1}} \leq \frac{C}{|\xi|}.$$

Estimate (4.8) then follows.

Case 2: $|\xi| \geq \frac{4\|d\|_\infty}{\varepsilon}$. Choose $\kappa > 0$ such that

$$C_0\kappa^2 \leq \|d\|_\infty \quad \text{and} \quad \|\nabla d\|_\infty \kappa \leq \|d\|_\infty.$$

Then, for $|z| \leq \kappa\varepsilon^{-1}$, we have that $|\nabla d(t,x)\cdot z| \leq |\xi|/4$ and $C_0\varepsilon|z|^2 \leq |\xi|/4$. In particular, (8.4) holds true. We write

$$\begin{aligned} a_\varepsilon &= \int_{\mathbb{R}^n} \left(\phi\left(\xi + \frac{d(t,x+\varepsilon z) - d(t,x)}{\varepsilon}\right) - \phi\left(\xi + \nabla d(t,x)\cdot z\right)\right)\frac{dz}{|z|^{n+1}} \\ &= \int_{\{|z|<\kappa\varepsilon^{-1}\}} (\dots) + \int_{\{|z|>\kappa\varepsilon^{-1}\}} (\dots) \\ &=: I + II. \end{aligned}$$

The same computations as in Case 1 give

$$|I| \leq \frac{C\varepsilon}{|\xi|^2}\int_{\{|z|<\kappa\varepsilon^{-1}\}} \frac{dz}{|z|^{n-1}} \leq \frac{C}{|\xi|^2}.$$

Next, we write

$$\begin{aligned} II &= \int_{\{|z|>\kappa\varepsilon^{-1}\}} \left(\phi\left(\xi + \frac{d(t,x+\varepsilon z) - d(t,x)}{\varepsilon}\right) - \phi\left(\xi\right)\right)\frac{dz}{|z|^{n+1}} \\ &\quad + \int_{\{|z|>\kappa\varepsilon^{-1}\}} \left(\phi\left(\xi + \nabla d(t,x)\cdot z\right) - \phi(\xi)\right)\frac{dz}{|z|^{n+1}} \\ &= J_1 + J_2. \end{aligned}$$

Since $|\xi| \geq 4\|d\|_\infty/\varepsilon$, we have that $|\xi + \frac{d(t,x+\varepsilon z)-d(t,x)}{\varepsilon}| \geq |\xi|/2$. Therefore, recalling that H is the Heaviside function, $H(\xi) = H\left(\xi + \frac{d(t,x+\varepsilon z)-d(t,x)}{\varepsilon}\right)$ and by (4.2)

$$\left|\phi\left(\xi + \frac{d(t,x+\varepsilon z) - d(t,x)}{\varepsilon}\right) - H\left(\xi + \frac{d(t,x+\varepsilon z) - d(t,x)}{\varepsilon}\right)\right|,\ |\phi\left(\xi\right) - H(\xi)| \leq \frac{C}{|\xi|},$$

from which it holds that

$$|J_1| \leq \frac{C}{|\xi|}\int_{\{|z|>\kappa\varepsilon^{-1}\}} \frac{dz}{|z|^{n+1}} = \frac{C\varepsilon}{|\xi|}. \tag{8.5}$$

Next, to estimate J_2 we write

$$J_2 = \mathrm{P.\,V.}\int_{\mathbb{R}^n} (\phi(\xi+\nabla d(t,x)\cdot z)-\phi(\xi))\,\frac{dz}{|z|^{n+1}} \tag{8.6}$$
$$- \mathrm{P.\,V.}\int_{\{|z|<\kappa\varepsilon^{-1}\}} (\phi(\xi+\nabla d(t,x)\cdot z)-\phi(\xi))\,\frac{dz}{|z|^{n+1}}.$$

By Lemma 3.2, equation (1.4) for ϕ, the periodicity of W and the fact that $W'(0)=0$,

$$\begin{aligned}\mathrm{P.\,V.}\int_{\mathbb{R}^n} (\phi(\xi+\nabla d(t,x)\cdot z)-\phi(\xi))\,\frac{dz}{|z|^{n+1}} &= |\nabla d(t,x)|C_n\mathcal{I}_1[\phi](\xi)\\ &= |\nabla d(t,x)|W'(\phi(\xi))\\ &= |\nabla d(t,x)|W'(\phi(\xi)-H(\xi))\\ &= O(\phi(\xi)-H(\xi)),\end{aligned}$$

with H the Heaviside function. In particular, by (4.2),

$$\left|\mathrm{P.\,V.}\int_{\mathbb{R}^n} (\phi(\xi+\nabla d(t,x)\cdot z)-\phi(\xi))\,\frac{dz}{|z|^{n+1}}\right| \le \frac{C}{|\xi|}. \tag{8.7}$$

Finally, we write

$$\begin{aligned}&\mathrm{P.\,V.}\int_{\{|z|<\kappa\varepsilon^{-1}\}} (\phi(\xi+\nabla d(t,x)\cdot z)-\phi(\xi))\,\frac{dz}{|z|^{n+1}}\\ &\quad= \int_{\{|z|<\kappa\varepsilon^{-1}\}} \left(\phi(\xi+\nabla d(t,x)\cdot z)-\phi(\xi)-\phi'(\xi)\nabla d(t,x)\cdot z\right)\frac{dz}{|z|^{n+1}}\end{aligned}$$

and since $|\nabla d(t,x)\cdot z|\le|\xi|/4$ for $|z|<\kappa\varepsilon^{-1}$, from (4.3), we estimate

$$\left|\phi(\xi+\nabla d(t,x)\cdot z)-\phi(\xi)-\phi'(\xi)\nabla d(t,x)\cdot z\right| \le \frac{C}{\xi^2}|z|^2.$$

Therefore,

$$\begin{aligned}\left|\mathrm{P.\,V.}\int_{\{|z|<\kappa\varepsilon^{-1}\}} (\phi(\xi+\nabla d(t,x)\cdot z)-\phi(\xi))\,\frac{dz}{|z|^{n+1}}\right| &\le \frac{C}{|\xi|^2}\int_{\{|z|<\kappa\varepsilon^{-1}\}}\frac{dz}{|z|^{n-1}}\\ &= \frac{C\varepsilon^{-1}}{|\xi|^2}\\ &\le \frac{C}{|\xi|},\end{aligned}$$

where in the last inequality we used that $\varepsilon^{-1}\le|\xi|/(4\|d\|_\infty)$. With this, (8.6), and (8.7), we infer that

$$|J_2|\le\frac{C}{|\xi|}$$

which with together with (8.5) yields

$$|II|\le\frac{C}{|\xi|}.$$

Estimate (4.8) for a_ε then follows. The estimate for $\dot{a}_\varepsilon$ is proven in the same way, just replacing ϕ by $\dot{\phi}$.

The proof of Lemma 4.2 is then completed. □

9. Proof of Lemma 4.6

Lemma 4.6 follows from the next three lemmas.

Lemma 9.1. *Let d be as in Definition 4.3. If $|d(t,x)| > \rho$, then there is a constant $C = C(n,\phi,d) > 0$ such that*

$$\left| a_\varepsilon \left(\frac{d(t,x)}{\varepsilon}; t, x \right) \right| \le \frac{C\varepsilon}{\rho}.$$

Lemma 9.2. *Let d be as in Definition 4.3. If $|d(t,x)| > \rho$, then there is a constant $C = C(n,\phi,d) > 0$ such that*

$$\left| \mathcal{I}_n \left[\phi \left(\frac{d(t,\cdot)}{\varepsilon} \right) \right](x) \right| \le \frac{C}{\rho}.$$

Lemma 9.3. *Let d be as in Definition 4.3. If $|d(t,x)| > \rho$, then there is a constant $C = C(n,\phi,d) > 0$ such that*

$$\left| \mathcal{I}_1[\phi] \left(\frac{d(t,x)}{\varepsilon} \right) \right| \le \frac{C\varepsilon}{\rho}.$$

9.1. **Proof of Lemma 9.1.** The lemma is an immediate consequence of (4.8).

9.2. **Proof of Lemma 9.2.** We drop the notation in t and begin by writing

$$\begin{aligned}
&\mathcal{I}_n \left[\phi \left(\frac{d(\cdot)}{\varepsilon} \right) \right](x) \\
&= \mathrm{P.V.} \int_{\mathbb{R}^n} \left(\phi \left(\frac{d(x+z)}{\varepsilon} \right) - \phi \left(\frac{d(x)}{\varepsilon} \right) \right) \frac{dz}{|z|^{n+1}} \\
&= \int_{|z|<\frac{\rho}{2}} \left(\phi \left(\frac{d(x+z)}{\varepsilon} \right) - \phi \left(\frac{d(x)}{\varepsilon} \right) - \dot{\phi} \left(\frac{d(x)}{\varepsilon} \right) \frac{\nabla d(x)}{\varepsilon} \cdot z \right) \frac{dz}{|z|^{n+1}} \\
&\quad + \int_{|z|>\frac{\rho}{2}} \left(\phi \left(\frac{d(x+z)}{\varepsilon} \right) - \phi \left(\frac{d(x)}{\varepsilon} \right) \right) \frac{dz}{|z|^{n+1}} \\
&=: I + II.
\end{aligned}$$

Looking first at the long-range interactions, we use $0 < \phi < 1$ and Lemma 3.1 to estimate

$$|II| \le 2 \int_{|z|>\frac{\rho}{2}} \frac{dz}{|z|^{n+1}} = \frac{C}{\rho}.$$

For the short-range interactions, fix z such that $|z| < \rho/2$. Then, by Taylor's theorem,

$$\begin{aligned}
&\phi \left(\frac{d(x+z)}{\varepsilon} \right) - \phi \left(\frac{d(x)}{\varepsilon} \right) - \dot{\phi} \left(\frac{d(x)}{\varepsilon} \right) \frac{\nabla d(x)}{\varepsilon} \cdot z \\
&= \frac{1}{2} \left(\ddot{\phi} \left(\frac{d(x+\tau z)}{\varepsilon} \right) \frac{\nabla d(x+\tau z) \otimes \nabla d(x+\tau z)}{\varepsilon^2} + \dot{\phi} \left(\frac{d(x+\tau z)}{\varepsilon} \right) \frac{D^2 d(x+\tau z)}{\varepsilon} \right) z \cdot z
\end{aligned}$$

for some $0 \le \tau \le 1$. Since

$$\left| \frac{d(x+\tau z)}{\varepsilon} \right| \ge \frac{|d(x)|}{\varepsilon} - \frac{|z|}{\varepsilon} > \frac{\rho}{\varepsilon} - \frac{\rho}{2\varepsilon} = \frac{\rho}{2\varepsilon},$$

we can apply (4.3) to estimate

$$\left| \dot{\phi} \left(\frac{d(x+\tau z)}{\varepsilon} \right) \right| \le \frac{C\varepsilon^2}{\rho^2} \quad \text{and} \quad \left| \ddot{\phi} \left(\frac{d(x+\tau z)}{\varepsilon} \right) \right| \le \frac{C\varepsilon^2}{\rho^2}.$$

Therefore, using that the first and second derivatives of d are bounded,

$$\left|\phi\left(\frac{d(x+z)}{\varepsilon}\right)-\phi\left(\frac{d(x)}{\varepsilon}\right)-\dot{\phi}\left(\frac{d(x)}{\varepsilon}\right)\frac{\nabla d(x)}{\varepsilon}\cdot z\right|\leq\frac{C}{\rho^2}\left|z\right|^2.$$

Thus, from Lemma 3.1, we have that

$$|I|\leq\frac{C}{\rho^2}\int_{|z|<\frac{\rho}{2}}|z|^2\frac{dz}{|z|^{n+1}}=\frac{C}{\rho}.$$

The conclusion follows by combining the estimates for I and II.

9.3. **Proof of Lemma 9.3.** From (1.4), the periodicity of W and $W'(0)=0$, we see that

$$\mathcal{I}_1[\phi]\left(\frac{d(x)}{\varepsilon}\right)=\frac{1}{C_n}W'\left(\phi\left(\frac{d(x)}{\varepsilon}\right)\right)=\frac{1}{C_n}W'\left(\tilde{\phi}\left(\frac{d(x)}{\varepsilon}\right)\right)=O\left(\tilde{\phi}\left(\frac{d(x)}{\varepsilon}\right)\right),$$

where, as usual, $\tilde{\phi}(\xi)=\phi(\xi)-H(\xi)$ and H is the Heaviside function. Therefore, by (4.2),

$$\left|\mathcal{I}_1[\phi]\left(\frac{d(x)}{\varepsilon}\right)\right|\leq\frac{C\varepsilon}{d(x)}\leq\frac{C\varepsilon}{\rho}.$$

The proof of Lemma 4.6 is then completed. ☐

10. Proof of Lemma 4.10

Lemma 4.10 is a consequence of the following three lemmas.

Lemma 10.1. *Let d be as in Definition 4.3. If $|d(t,x)|<\rho$, then there is a constant $C=C(n,\phi,d)>0$ such that*

$$\left|\varepsilon\mathcal{I}_n\left[\psi\left(\frac{d(t,\cdot)}{\varepsilon};t,\cdot\right)\right](x)-C_n\mathcal{I}_1[\psi\left(\cdot;t,x\right)]\left(\frac{d(t,x)}{\varepsilon}\right)\right|\leq\frac{C}{|\ln\varepsilon|}.$$

Lemma 10.2. *Let d be as in Definition 4.3. If $|d(t,x)|>\rho$, then there is a constant $C=C(n,\phi,d)>0$ such that*

$$\left|\mathcal{I}_1[\psi\left(\cdot;t,x\right)]\left(\frac{d(t,x)}{\varepsilon}\right)\right|\leq\frac{C}{|\ln\varepsilon|\rho}.$$

Lemma 10.3. *Let d be as in Definition 4.3. If $|d(t,x)|>\rho$, then there is a constant $C=C(n,\phi,d)>0$ such that*

$$\left|\varepsilon\mathcal{I}_n\left[\psi\left(\frac{d(t,\cdot)}{\varepsilon};t,\cdot\right)\right](x)\right|\leq\frac{C}{|\ln\varepsilon|\rho}.$$

10.1. **Proof of Lemma 10.1.** For simplicity in notation, we drop the dependence on t. By Lemma 3.2 and the fact that $|\nabla d(x)|=1$, we have

$$\begin{aligned}
&\varepsilon\mathcal{I}_n\left[\psi\left(\frac{d(\cdot)}{\varepsilon};\cdot\right)\right](x)-C_n\mathcal{I}_1[\psi\left(\cdot;x\right)]\left(\frac{d(x)}{\varepsilon}\right)\\
&=P.V.\int_{\mathbb{R}^n}\left(\psi\left(\frac{d(x+\varepsilon z)}{\varepsilon};x+\varepsilon z\right)-\psi\left(\frac{d(x)}{\varepsilon}+\nabla d(x)\cdot z;x\right)\right)\frac{dz}{|z|^{n+1}}\\
&=P.V.\int_{\mathbb{R}^n}\left(\psi\left(\frac{d(x+\varepsilon z)}{\varepsilon};x\right)-\psi\left(\frac{d(x)}{\varepsilon}+\nabla d(x)\cdot z;x\right)\right)\frac{dz}{|z|^{n+1}}\\
&\quad+P.V.\int_{\mathbb{R}^n}\left(\psi\left(\frac{d(x+\varepsilon z)}{\varepsilon};x+\varepsilon z\right)-\psi\left(\frac{d(x+\varepsilon z)}{\varepsilon};x\right)\right)\frac{dz}{|z|^{n+1}}\\
&=:I+II.
\end{aligned}$$

First, let us estimate I. We will show that

$$|I| \le \frac{C}{|\ln\varepsilon|}. \tag{10.1}$$

Notice that

$$\left|\psi\left(\frac{d(x+\varepsilon z)}{\varepsilon};x\right)-\psi\left(\frac{d(x)}{\varepsilon}+\nabla d(x)\cdot z;x\right)\right| \le C\|\dot\psi\|_\infty \varepsilon |z|^2, \tag{10.2}$$

so that we can actually write

$$I=\int_{\mathbb{R}^n}\left(\psi\left(\frac{d(x+\varepsilon z)}{\varepsilon};x\right)-\psi\left(\frac{d(x)}{\varepsilon}+\nabla d(x)\cdot z;x\right)\right)\frac{dz}{|z|^{n+1}}.$$

We write

$$\begin{aligned} I &= \int_{\{|z|<\varepsilon^{-\frac12}\}}(\dots)+\int_{\{|z|>\varepsilon^{-\frac12}\}}(\dots)\\ &=: I_1+I_2. \end{aligned}$$

Recalling Lemma 3.1, by (10.2) and (4.17) for $\dot\psi$

$$|I_1|\le C\varepsilon\|\dot\psi\|_\infty\int_{\{|z|<\varepsilon^{-\frac12}\}}\frac{dz}{|z|^{n-1}}\le \frac{C\varepsilon}{\varepsilon^{\frac12}|\ln\varepsilon|}\varepsilon^{-\frac12}=\frac{C}{|\ln\varepsilon|},$$

and from (4.17) for ψ,

$$|I_2|\le 2\|\psi\|_\infty\int_{\{|z|>\varepsilon^{-\frac12}\}}\frac{dz}{|z|^{n+1}}\le C\frac{\varepsilon^{\frac12}}{\varepsilon^{\frac12}|\ln\varepsilon|}=\frac{C}{|\ln\varepsilon|}.$$

This proves (10.1).

Next, using that $\mathrm{P.V.}\int_{\{|z|<\varepsilon^{-\frac12}\}}\nabla_x\psi\left(\frac{d(x)}{\varepsilon};x\right)\cdot z\frac{dz}{|z|^{n+1}}=0$, we write

$$\begin{aligned} II &= \Bigg[\int_{\{|z|<\varepsilon^{-\frac12}\}}\left(\psi\left(\frac{d(x+\varepsilon z)}{\varepsilon};x+\varepsilon z\right)-\psi\left(\frac{d(x+\varepsilon z)}{\varepsilon};x\right)\right.\\ &\quad\left.-\nabla_x\psi\left(\frac{d(x+\varepsilon z)}{\varepsilon};x\right)\cdot(\varepsilon z)\right)\frac{dz}{|z|^{n+1}}\Bigg]\\ &\quad+\left[\int_{\{|z|<\varepsilon^{-\frac12}\}}\left(\nabla_x\psi\left(\frac{d(x+\varepsilon z)}{\varepsilon};x\right)-\nabla_x\psi\left(\frac{d(x)}{\varepsilon};x\right)\right)\cdot(\varepsilon z)\frac{dz}{|z|^{n+1}}\right]\\ &\quad+\left[\int_{\{|z|>\varepsilon^{-\frac12}\}}\left(\psi\left(\frac{d(x+\varepsilon z)}{\varepsilon};x+\varepsilon z\right)-\psi\left(\frac{d(x+\varepsilon z)}{\varepsilon};x\right)\right)\frac{dz}{|z|^{n+1}}\right]\\ &= II_1+II_2+II_3. \end{aligned}$$

Again, recalling Lemma 3.1, by (4.17) for $D_x^2\psi$,

$$|II_1|\le C\|D_x^2\psi\|_\infty\varepsilon^2\int_{|z|<\varepsilon^{-\frac12}}\frac{dz}{|z|^{n-1}}\le\frac{C\varepsilon}{|\ln\varepsilon|}.$$

By (4.17) for $\nabla_x\dot\psi$,

$$|II_2|\le C\|\nabla_x\dot\psi\|_\infty\varepsilon\int_{|z|<\varepsilon^{-\frac12}}\frac{dz}{|z|^{n-1}}\le\frac{C}{|\ln\varepsilon|}.$$

Finally, from (4.17) for ψ,

$$|II_3| \le 2\|\psi\|_\infty \int_{|z|>\varepsilon^{-\frac{1}{2}}} \frac{dz}{|z|^{n+1}} \le \frac{C}{|\ln\varepsilon|}.$$

This proves that

$$|II| \le \frac{C}{|\ln\varepsilon|}. \tag{10.3}$$

From (10.1) and (10.3) estimate (10.1) follows.

10.2. **Proof of Lemma 10.2.** From (4.13),(4.2), (4.3), (4.8), (4.10) and (4.18), for $|\xi| \ge 1$,

$$\begin{aligned}|\mathcal{I}_1[\psi(\cdot;t,x)](\xi)| &\le C|\psi(\xi)| + \frac{|a_\varepsilon(\xi;t,x)|}{\varepsilon|\ln\varepsilon|} + C|1+\bar{a}_\varepsilon(t,x)|\dot{\phi}(\xi) + C|\phi(\xi)| \\ &\le \frac{C}{\varepsilon|\ln\varepsilon||\xi|}.\end{aligned}$$

In particular,

$$\left|\mathcal{I}_1[\psi(\cdot;t,x)]\left(\frac{d(t,x)}{\varepsilon}\right)\right| \le \frac{C}{|\ln\varepsilon||d(t,x)|} \le \frac{C}{|\ln\varepsilon|\rho}.$$

10.3. **Proof of Lemma 10.3.** For convenience, we drop the dependence on t and begin by writing

$$\begin{aligned}\varepsilon\mathcal{I}_n\left[\psi\left(\frac{d(\cdot)}{\varepsilon};\cdot\right)\right](x) = \varepsilon\mathcal{I}_n&\left[\psi\left(\frac{d(\cdot)}{\varepsilon};\cdot\right)\right](x) - |\nabla d(x)|C_n\mathcal{I}_1[\psi(\cdot;x)] \\ &+ |\nabla d(x)|C_n\mathcal{I}_1[\psi(\cdot;x)]\end{aligned}$$

By Lemma 3.2,

$$\begin{aligned}&\varepsilon\mathcal{I}_n\left[\psi\left(\frac{d(\cdot)}{\varepsilon};\cdot\right)\right](x) - |\nabla d(x)|C_n\mathcal{I}_1[\psi(\cdot;x)] \\ &= \int_{\mathbb{R}^n}\left(\psi\left(\frac{d(x+\varepsilon z)}{\varepsilon};x+\varepsilon z\right) - \psi\left(\frac{d(x)}{\varepsilon}+\nabla d(x)\cdot z;x\right)\right)\frac{dz}{|z|^{n+1}}.\end{aligned}$$

Therefore, as in the proof of Lemma 10.1, we obtain

$$\left|\mathcal{I}_n\left[\psi\left(\frac{d(\cdot)}{\varepsilon};\cdot\right)\right](x) - |\nabla d(x)|C_n\mathcal{I}_1[\psi(\cdot;x)]\right| \le \frac{C}{|\ln\varepsilon|\rho}.$$

Moreover, by Lemma 10.2,

$$|\nabla d(x)|\,|C_n\mathcal{I}_1[\psi(\cdot;x)]| \le \frac{C}{|\ln\varepsilon|\rho}.$$

The lemma then follows.

The proof of Lemma 4.10 is then completed. □

11. Proof of Lemma 4.4

For simplicity, we drop the dependence of d, a_ε and $\bar{a}_\varepsilon$ on t.

Recall from Definition 4.3 that in Q_ρ, the function d is the smooth signed distance function from a smooth front. Therefore, in Q_ρ the eigenvalues of $D^2d(x)$ are

$$\lambda_i(x) = \frac{-\kappa_i}{1-\kappa_i d(x)},\ i = 1,\dots,n-1, \quad \lambda_n(x) = 0,$$

see, e.g., [15, Lemma 14.17]. Moreover, since $|\nabla d| = 1$ in Q_ρ, we have the equation $D^2 d(x)\nabla d(x) = 0$ from which we see that $\nabla d(x)$ is an eigenvector, with norm 1, for $D^2 d(x)$ with associated eigenvalue $\lambda_n(x) = 0$.

Given $0 < r < 1$, if $|z| < r/\varepsilon$,

$$\frac{d(x+\varepsilon z) - d(x)}{\varepsilon} = \nabla d(x)\cdot z + \frac{\varepsilon}{2} D^2 d(x) z\cdot z + O(r\varepsilon|z|^2). \tag{11.1}$$

Let $T = (v_1, \dots, v_n)$ be an orthogonal matrix whose columns are a set of orthonormal eigenvectors $v_1, \dots, v_n$ for the eigenvalues $\lambda_1, \dots, \lambda_n$ with $v_n = \nabla d(x)$. Then, if $y = Tz$, the right-hand side of (11.1) can be written as

$$\nabla d(x)\cdot z + \frac{\varepsilon}{2} D^2 d(x) z\cdot z + O(r\varepsilon|z|^2) = y_n + \frac{\varepsilon}{2}\sum_{i=1}^{n-1}\lambda_i y_i^2 + O(r\varepsilon|y|^2). \tag{11.2}$$

Denote $z = (z', z_n)$ with $z' \in \mathbb{R}^{n-1}$ and split $a_\varepsilon = a_\varepsilon(\xi; x)$ as follows, for $r > 0$,

$$\begin{aligned} a_\varepsilon &= \int_{\mathbb{R}^n}\left(\phi\left(\xi + \frac{d(x+\varepsilon z)-d(x)}{\varepsilon}\right) - \phi\left(\xi + \nabla d(x)\cdot z\right)\right)\frac{dz}{|z|^{n+1}} \\ &= \int_{\{|z'|,\,|z_n|<1\}}(\dots) + \int_{\{1<|z_n|<\frac{r}{\varepsilon},\,|z'|<1\}\cup\{1<|z'|<\frac{r}{\varepsilon},\,|z_n|<\frac{r}{\varepsilon},\}}(\dots) + \int_{\{|z'|>\frac{r}{\varepsilon}\}\cup\{|z_n|>\frac{r}{\varepsilon}\}}(\dots). \end{aligned}$$

By the regularity of ϕ and d and by Lemma 3.1, we see that

$$\begin{aligned} &\left|\int_{\{|z'|,\,|z_n|<1\}}\left(\phi\left(\xi + \frac{d(x+\varepsilon z)-d(x)}{\varepsilon}\right) - \phi\left(\xi + \nabla d(x)\cdot z\right)\right)\frac{dz}{|z|^{n+1}}\right| \\ &\le C\varepsilon\int_{\{|z|<\sqrt{2}\}}\frac{dz}{|z|^{n-1}} \le C\varepsilon, \end{aligned}$$

and using that $0 < \phi < 1$,

$$\begin{aligned} &\left|\int_{\{|z'|>\frac{r}{\varepsilon}\}\cup\{|z_n|>\frac{r}{\varepsilon}\}}\left(\phi\left(\xi + \frac{d(x+\varepsilon z)-d(x)}{\varepsilon}\right) - \phi\left(\xi + \nabla d(x)\cdot z\right)\right)\frac{dz}{|z|^{n+1}}\right| \\ &\le 2\int_{\{|z|>\frac{r}{\varepsilon}\}}\frac{dz}{|z|^{n+1}} \le C\frac{\varepsilon}{r}. \end{aligned}$$

Recalling that $\phi(-\infty) = 0$ and $\phi(\infty) = 1$, the previous inequalities yield

$$\begin{aligned} \bar a_\varepsilon(x) &= \frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}} a_\varepsilon(\xi; x)\,\dot\phi(\xi)\,d\xi \\ &= \frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}}\dot\phi(\xi) I(\xi)\,d\xi + \frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}}\dot\phi(\xi) II(\xi)\,d\xi + O(|\ln(\varepsilon)|^{-1}) + O(r^{-1}|\ln(\varepsilon)|^{-1}) \end{aligned} \tag{11.3}$$

where

$$\begin{aligned} I(\xi) &:= \frac{1}{\varepsilon}\int_{\{1<|z'|<\frac{r}{\varepsilon},\,|z_n|<\frac{r}{\varepsilon}\}}\left(\phi\left(\xi + \frac{d(x+\varepsilon z)-d(x)}{\varepsilon}\right) - \phi\left(\xi + \nabla d(x)\cdot z\right)\right)\frac{dz}{|z|^{n+1}} \\ II(\xi) &:= \frac{1}{\varepsilon}\int_{\{|z'|<1,\,1<|z_n|<\frac{r}{\varepsilon}\}}\left(\phi\left(\xi + \frac{d(x+\varepsilon z)-d(x)}{\varepsilon}\right) - \phi\left(\xi + \nabla d(x)\cdot z\right)\right)\frac{dz}{|z|^{n+1}}. \end{aligned}$$

We will estimate $\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot\phi(\xi) I(\xi)\,d\xi$ and $\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot\phi(\xi) II(\xi)\,d\xi$ separately.

Step 1. Estimating $\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot\phi(\xi)I(\xi)\,d\xi$. Let us start by estimating $I(\xi)$. Recalling (11.2), with the change of variables, $\varepsilon z = Ty$, we write $I(\xi)$ as

$$I(\xi)=\int_{\{\varepsilon<|y'|<r,\,|y_n|<r\}}\left(\phi\left(\xi+\frac{1}{\varepsilon}\left(y_n+\frac{1}{2}\sum_{i=1}^{n-1}\lambda_i y_i^2+O(r|y|^2)\right)\right)-\phi\left(\xi+\frac{y_n}{\varepsilon}\right)\right)\frac{dy}{|y|^{n+1}}.$$

Let us denote

$$A(y'):=\frac{1}{2}\sum_{i=1}^{n-1}\lambda_i y_i^2, \tag{11.4}$$

then, with the further change of variable $t=y_n/|y'|$, we write

$$\begin{aligned}I(\xi)&=\int_{\{\varepsilon<|y'|<r\}}\frac{dy'}{|y'|^{n+1}}\int_{\{|y_n|<r\}}\frac{dy_n}{\left(\frac{y_n^2}{|y'|^2}+1\right)^{\frac{n+1}{2}}}\\&\qquad\left\{\phi\left(\xi+\frac{1}{\varepsilon}\left(y_n+A(y')+O(r|y|^2)\right)\right)-\phi\left(\xi+\frac{y_n}{\varepsilon}\right)\right\}\\&=\int_{\{\varepsilon<|y'|<r\}}\frac{dy'}{|y'|^{n}}\int_{\{|t|<\frac{r}{|y'|}\}}\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\\&\qquad\left\{\phi\left(\xi+\frac{|y'|}{\varepsilon}\left[t+|y'|A\left(\frac{y'}{|y'|}\right)+|y'|O\left(r(1+t^2)\right)\right]\right)-\phi\left(\xi+\frac{t|y'|}{\varepsilon}\right)\right\}.\end{aligned}$$

Define $b(\theta,r,t)$ by

$$b(\theta,t,r):=A(\theta)+O\left(r(1+t^2)\right). \tag{11.5}$$

Note that if $|t|<1$, then for some $C_0>0$,

$$|b(\theta,r,t)-A(\theta)|\le C_0 r. \tag{11.6}$$

Then, using polar coordinates $y'=\rho\theta$, $\rho>0$ and $\theta\in S^{n-2}$, we can write $I(\xi)$ as

$$I(\xi)=\int_\varepsilon^r\frac{d\rho}{\rho^2}\int_{S^{n-2}}d\theta\int_{\{|t|<\frac{r}{\rho}\}}\left\{\phi\left(\xi+\frac{\rho}{\varepsilon}\left(t+\rho b(\theta,t,r)\right)\right)-\phi\left(\xi+\frac{t\rho}{\varepsilon}\right)\right\}\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}.$$

We will prove that one of the main contributions in the integral above comes from values of ρ between $\varepsilon^{\frac12}$ and r and values of t such that, for $A(\theta)>0$,

$$-\rho A(\theta)<t<0, \tag{11.7}$$

and for $A(\theta)<0$,

$$0<t<-\rho A(\theta). \tag{11.8}$$

Indeed, by (4.2) if $A(\theta)>0$, for points t as in (11.7), the integrand function in $I(\xi)$ is close to 1 and thus

$$\begin{aligned}\frac{1}{|\ln(\varepsilon)|}\int_{\mathbb{R}}d\xi\dot\phi(\xi)\int_{\varepsilon^{\frac12}}^r\frac{d\rho}{\rho^2}\int_{-\rho A(\theta)}^0(\dots)dt&\simeq\frac{1}{|\ln(\varepsilon)|}\int_{\mathbb{R}}d\xi\dot\phi(\xi)\int_{\varepsilon^{\frac12}}^r\frac{d\rho}{\rho^2}\int_{-\rho A(\theta)}^0dt\\&=\frac{A(\theta)}{|\ln(\varepsilon)|}\int_{\mathbb{R}}d\xi\dot\phi(\xi)\int_{\varepsilon^{\frac12}}^r\frac{d\rho}{\rho}\\&\simeq\frac{A(\theta)}{2},\end{aligned}$$

see (11.15) below. Similarly, if $A(\theta)<0$, for points t as in (11.8) the integrand function is close to -1 and

$$\frac{1}{|\ln(\varepsilon)|}\int_{\mathbb{R}} d\xi\dot{\phi}(\xi)\int_{\varepsilon^{\frac12}}^{r}\frac{d\rho}{\rho^2}\int_0^{-\rho A(\theta)}(\dots)dt\simeq\frac{A(\theta)}{2}.$$

The other main contributions come from values of ρ between ε and $\varepsilon^{\frac12}$ and values of t such that, for $A(\theta)>0$,

$$-1<t<-\rho A(\theta)\quad\text{or}\quad 0<t<1,$$

and for $A(\theta)<0$

$$-1<t<0\quad\text{or}\quad <-\rho A(\theta)<t<1.$$

Indeed, we will show that for $A(\theta)>0$,

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} d\xi\dot{\phi}(\xi)\int_{-1}^{r}\frac{d\rho}{\rho^2}\int_{-1}^{-\rho A(\theta)}(\dots)dt\simeq A(\theta)\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\frac12\frac{d}{d\xi}(\phi(\xi))^2d\xi\int_{\varepsilon}^{\varepsilon^{\frac12}}\frac{d\rho}{\rho}=\frac{A(\theta)}{4},$$

and

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} d\xi\dot{\phi}(\xi)\int_{-1}^{r}\frac{d\rho}{\rho^2}\int_0^1(\dots)dt\simeq\frac{A(\theta)}{4},$$

see (11.24) and (11.25). A similar estimate holds when $A(\theta)<0$.

To formally prove the estimates above, we start by splitting $I(\xi)$ as

$$\begin{aligned}I(\xi)&=\int_\varepsilon^r\frac{d\rho}{\rho^2}\int_{S^{n-2}}d\theta\int_{\{|t|<1\}}dt(\dots)+\int_\varepsilon^r\frac{d\rho}{\rho^2}\int_{S^{n-2}}d\theta\int_{\{1<|t|<\frac{r}{\rho}\}}dt(\dots)\\&=:I_1(\xi)+I_2(\xi)\end{aligned}\tag{11.9}$$

and then estimate $\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot{\phi}(\xi)I_1(\xi)\,d\xi$ and $\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot{\phi}(\xi)I_2(\xi)\,d\xi$ separately.

Step 1a. Estimating $\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot{\phi}(\xi)I_1(\xi)\,d\xi$. We will show that

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot{\phi}(\xi)I_1(\xi)\,d\xi=\int_{S^{n-2}}A(\theta)d\theta+o_\varepsilon(1)+o_\delta(1)+O(r),\tag{11.10}$$

where $o_\varepsilon(1)$ depends on the parameters δ and r. For $C_0r<\delta<1$ with C_0 as in (11.6), we write

(11.11)
$$\begin{aligned}I_1(\xi)&=\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta(\dots)+\int_{S^{n-2}\cap\{A(\theta)<-3\delta\}}d\theta(\dots)+\int_{S^{n-2}\cap\{|A(\theta)|\le3\delta\}}d\theta(\dots)\\&=:I_1^1(\xi)+I_1^2(\xi)+I_1^3(\xi).\end{aligned}$$

Beginning with $I_1^1(\xi)$, we further split

$$\begin{aligned}I_1^1(\xi)=&\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_\varepsilon^r\frac{d\rho}{\rho^2}\int_{-1}^{-\rho(A(\theta)+2\delta)}dt(\dots)\\&+\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_\varepsilon^r\frac{d\rho}{\rho^2}\int_{-\rho(A(\theta)+2\delta)}^{-\rho(A(\theta)-2\delta)}dt(\dots)\\&+\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_\varepsilon^r\frac{d\rho}{\rho^2}\int_{-\rho(A(\theta)-2\delta)}^{-\delta\rho}dt(\dots)\\&+\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_\varepsilon^r\frac{d\rho}{\rho^2}\int_{-\delta\rho}^{\delta\rho}dt(\dots)\end{aligned}$$

$$+\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_\varepsilon^r \frac{d\rho}{\rho^2}\int_{\delta\rho}^1 dt(\dots)$$
$$=: J_1(\xi)+J_2(\xi)+J_3(\xi)+J_4(\xi)+J_5(\xi).$$

Notice that, for $0<\rho<r$ and $r>0$ small enough,

$$1>\rho(A(\theta)+\delta).$$

In what follows, we will use several times that, by (4.2), if $M>1$,

$$\int_{\{|\xi|<M\}}\dot\phi(\xi)d\xi=\phi(M)-\phi(-M)=1-\frac{1}{\alpha M}-\frac{1}{\alpha M}+O(M^{-2})=1+O(M^{-1}), \tag{11.12}$$

and

$$\int_{\{|\xi|>M\}}\dot\phi(\xi)d\xi=1-\int_{\{|\xi|<M\}}\dot\phi(\xi)d\xi=O(M^{-1}). \tag{11.13}$$

We will also use that

$$\phi(\xi+\eta_1)-\phi(\xi+\eta_2)=O(\eta_1-\eta_2)\quad\text{for all }\xi,\eta_1,\eta_2\in\mathbb{R}. \tag{11.14}$$

Moreover, recalling (11.6) and that $C_0r<\delta$, if $A(\theta)>3\delta$ and $|t|<1$, then $b(\theta,t,r)>0$ and by the monotonicity of ϕ,

$$\phi\left(\xi+\frac{\rho}{\varepsilon}\left(t+\rho b(\theta,t,r)\right)\right)-\phi\left(\xi+\frac{t\rho}{\varepsilon}\right)>0.$$

We claim that

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot\phi(\xi)J_3(\xi)d\xi=\frac12\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}A(\theta)d\theta+o_\varepsilon(1)+O(\delta)+O(r). \tag{11.15}$$

For $R>0$, we split

$$\begin{aligned}J_3(\xi)=&\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_\varepsilon^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-\rho(A(\theta)-2\delta)}^{-\delta\rho}dt(\dots)\\&+\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_{(R\varepsilon)^{\frac12}}^{r}\frac{d\rho}{\rho^2}\int_{-\rho(A(\theta)-2\delta)}^{-\delta\rho}dt(\dots).\end{aligned} \tag{11.16}$$

For the first integral on the right-hand side of (11.16), we use (11.14) to estimate

$$\begin{aligned}0&\le\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_\varepsilon^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-\rho(A(\theta)-2\delta)}^{-\delta\rho}dt\,(\dots)\\&\le C\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_\varepsilon^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-\rho(A(\theta)-2\delta)}^{-\delta\rho}\frac{\rho^2}{\varepsilon}b(\theta,r,t)\,dt\\&\le C\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_\varepsilon^{(R\varepsilon)^{\frac12}}\rho(A(\theta)-3\delta)\frac{d\rho}{\varepsilon}\\&\le C\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_\varepsilon^{(R\varepsilon)^{\frac12}}\frac{\rho}{\varepsilon}d\rho\le CR.\end{aligned}$$

Therefore, regarding the first integral in (11.16), we have

$$0\le\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot\phi(\xi)d\xi\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_\varepsilon^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-\rho(A(\theta)-2\delta)}^{-\delta\rho}dt(\dots)\le\frac{CR}{|\ln\varepsilon|}. \tag{11.17}$$

Next, we proceed with the estimate of the second integral in the right-hand side of (11.16). Notice that, for $R\delta > 2$ and $C_0 r < \delta < 1$ with C_0 as in (11.6), if

$$\theta \in S^{n-2} \cap \{A(\theta) > 3\delta\}, \quad (R\varepsilon)^{\frac{1}{2}} < \rho < r, \quad -\rho(A(\theta) - 2\delta) < t < -\rho\delta, \quad |\xi| < \frac{R\delta}{2},$$

then

$$\xi + \frac{t\rho}{\varepsilon} < \xi - \frac{\delta\rho^2}{\varepsilon} < \xi - R\delta < -\frac{R\delta}{2}$$

and

$$\xi + \frac{\rho}{\varepsilon}(t + \rho b(\theta, t, r)) > \xi + \frac{\rho}{\varepsilon}(t + \rho A(\theta) - \rho C_0 r) > \xi + \frac{\delta\rho^2}{\varepsilon} > \xi + R\delta > \frac{R\delta}{2}.$$

Consequently, by (4.2), if H is the Heaviside function,

$$\begin{aligned}
&\phi\left(\xi + \frac{\rho}{\varepsilon}\left(t + \rho b(\theta, t, r)\right)\right) - \phi\left(\xi + \frac{t\rho}{\varepsilon}\right) \\
&= H\left(\xi + \frac{\rho}{\varepsilon}\left(t + \rho b(\theta, t, r)\right)\right) - H\left(\xi + \frac{t\rho}{\varepsilon}\right) + O\left(\frac{1}{\xi + \frac{\rho}{\varepsilon}\left(t + \rho b(\theta, t, r)\right)}\right) + O\left(\frac{1}{\xi + \frac{t\rho}{\varepsilon}}\right) \\
&= 1 + O\left(\frac{1}{R\delta}\right).
\end{aligned}$$

Therefore,

(11.18)

$$\begin{aligned}
&\int_{\mathbb{R}} \dot\phi(\xi) d\xi \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac{1}{2}}}^{r} \frac{d\rho}{\rho^2} \int_{-\rho(A(\theta)-2\delta)}^{-\delta\rho} dt(\ldots) \\
&= \int_{-\frac{R\delta}{2}}^{\frac{R\delta}{2}} \dot\phi(\xi) d\xi \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac{1}{2}}}^{r} \frac{d\rho}{\rho^2} \int_{-\rho(A(\theta)-2\delta)}^{-\delta\rho} \left(1 + O\left(\frac{1}{R\delta}\right)\right) \frac{dt}{(t^2+1)^{\frac{n+1}{2}}} \\
&\quad + \int_{\{|\xi| > \frac{R\delta}{2}\}} \dot\phi(\xi) d\xi \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac{1}{2}}}^{r} \frac{d\rho}{\rho^2} \int_{-\rho(A(\theta)-2\delta)}^{-\delta\rho} dt(\ldots).
\end{aligned}$$

The main contribution in $\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} \dot\phi(\xi) J_3(\xi) d\xi$ comes from the integral of 1 in (11.18). Indeed, since $|t| < \rho(A(\theta) - 2\delta) < Cr$ implies

$$\frac{1}{(t^2+1)^{\frac{n+1}{2}}} = 1 + O(r),$$

we can write

$$\begin{aligned}
&\frac{1}{|\ln\varepsilon|} \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac{1}{2}}}^{r} \frac{d\rho}{\rho^2} \int_{-\rho(A(\theta)-2\delta)}^{-\rho\delta} \frac{dt}{(t^2+1)^{\frac{n+1}{2}}} \\
&= (1 + O(r)) \frac{1}{|\ln\varepsilon|} \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac{1}{2}}}^{r} \frac{1}{\rho^2} \rho(A(\theta) - 3\delta) d\rho \\
&= (1 + O(r)) \frac{1}{|\ln\varepsilon|} \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} (A(\theta) - 3\delta) d\theta \int_{(R\varepsilon)^{\frac{1}{2}}}^{r} \frac{1}{\rho} d\rho \\
&= (1 + O(r)) \frac{\ln(r) - \frac{1}{2}\ln(R) - \frac{1}{2}\ln(\varepsilon)}{|\ln\varepsilon|} \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} (A(\theta) - 3\delta) d\theta \\
&= \frac{1}{2} \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta) d\theta + o_\varepsilon(1) + O(\delta) + O(r).
\end{aligned}$$

With this and recalling (11.12), we infer that

$$\begin{aligned} &\frac{1}{|\ln\varepsilon|}\int_{-\frac{R\delta}{2}}^{\frac{R\delta}{2}}\dot\phi(\xi)d\xi\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_{(R\varepsilon)^{\frac12}}^{r}\frac{d\rho}{\rho^2}\int_{-\rho(A(\theta)-2\delta)}^{-\rho\delta}\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\\ &\quad=\frac12\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}A(\theta)d\theta+o_\varepsilon(1)+O(\delta)+O(r)+O\left(\frac{1}{R\delta}\right).\end{aligned}\tag{11.19}$$

Next, we look at the error terms in (11.18). First, note that

$$\begin{aligned} &\frac{1}{|\ln\varepsilon|}\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_{(R\varepsilon)^{\frac12}}^{r}\frac{d\rho}{\rho^2}\int_{-\rho(A(\theta)-2\delta)}^{-\delta\rho}\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\\ &\quad\le\frac{1}{|\ln\varepsilon|}\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_{(R\varepsilon)^{\frac12}}^{r}\frac{1}{\rho^2}\rho(A(\theta)-3\delta)d\rho\\ &\quad\le\frac{1}{|\ln\varepsilon|}\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_{(R\varepsilon)^{\frac12}}^{r}\frac{d\rho}{\rho}\\ &\quad\le C\frac{\ln(r)-\frac12\ln(R)-\frac12\ln(\varepsilon)}{|\ln\varepsilon|}\le C.\end{aligned}$$

With this, we estimate

$$\begin{aligned} &\left|\frac{1}{|\ln\varepsilon|}\int_{-\frac{R\delta}{2}}^{\frac{R\delta}{2}}\dot\phi(\xi)d\xi\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_{(R\varepsilon)^{\frac12}}^{r}\frac{d\rho}{\rho^2}\int_{-\rho(A(\theta)-2\delta)}^{-\delta\rho}O\left(\frac{1}{R\delta}\right)\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\right|\\ &\le O\left(\frac{1}{R\delta}\right),\end{aligned}\tag{11.20}$$

and similarly, using that $0<\phi<1$ and (11.13),

$$\begin{aligned} 0&\le\frac{1}{|\ln\varepsilon|}\int_{\{|\xi|>\frac{R\delta}{2}\}}\dot\phi(\xi)d\xi\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_{(R\varepsilon)^{\frac12}}^{r}\frac{d\rho}{\rho^2}\int_{-\rho(A(\theta)-2\delta)}^{-\delta\rho}dt(\dots)\\ &\le\frac{2}{|\ln\varepsilon|}\int_{\{|\xi|>\frac{R\delta}{2}\}}\dot\phi(\xi)d\xi\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_{(R\varepsilon)^{\frac12}}^{r}\frac{d\rho}{\rho^2}\int_{-\rho(A(\theta)-2\delta)}^{-\delta\rho}\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\\ &\le O\left(\frac{1}{R\delta}\right).\end{aligned}\tag{11.21}$$

Choosing

$$R=\delta^{-2},$$

from (11.16), (11.17), (11.19), (11.20) and (11.21), estimate (11.15) follows.

Next, let us estimate $\int_{\mathbb{R}}\dot\phi(\xi)J_2(\xi)d\xi$ and $\int_{\mathbb{R}}\dot\phi(\xi)J_4(\xi)d\xi$. Using that $0<\phi<1$, we get

$$\begin{aligned} 0&\le\int_{\mathbb{R}}\dot\phi(\xi)J_2(\xi)d\xi\le\int_{\mathbb{R}}\dot\phi(\xi)d\xi\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_\varepsilon^r\frac{d\rho}{\rho^2}\int_{-\rho(A(\theta)+2\delta)}^{-\rho(A(\theta)-2\delta)}dt\\ &\le\int_{\mathbb{R}}\dot\phi(\xi)d\xi\int_{S^{n-2}}d\theta\int_\varepsilon^r\frac{d\rho}{\rho^2}4\delta\rho=C\delta|\ln\varepsilon|,\end{aligned}$$

from which it follows that

$$\frac{1}{|\ln\varepsilon|}\left|\int_{\mathbb{R}}\dot\phi(\xi)J_2(\xi)d\xi\right|\le C\delta.\tag{11.22}$$

Similarly, we find that

$$\frac{1}{|\ln\varepsilon|}\left|\int_{\mathbb{R}}\dot{\phi}(\xi)J_4(\xi)d\xi\right|\leq C\delta. \tag{11.23}$$

Finally, let us estimate $\int_{\mathbb{R}}\dot{\phi}(\xi)J_1(\xi)d\xi$ and $\int_{\mathbb{R}}\dot{\phi}(\xi)J_5(\xi)d\xi$. We are going to show that

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot{\phi}(\xi)J_1(\xi)d\xi=\frac{1}{4}\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}A(\theta)d\theta+o_\varepsilon(1)+O(r)+O(\delta), \tag{11.24}$$

and

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot{\phi}(\xi)J_5(\xi)d\xi=\frac{1}{4}\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}A(\theta)d\theta+o_\varepsilon(1)+O(r)+O(\delta). \tag{11.25}$$

Beginning with J_1, for $R>0$, we split

$$\begin{aligned}J_1(\xi)&=\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_{\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-1}^{-\rho(A(\theta)+2\delta)}dt(\dots)\\&\quad+\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_{(R\varepsilon)^{\frac12}}^{r}\frac{d\rho}{\rho^2}\int_{-1}^{-\rho(A(\theta)+2\delta)}dt(\dots).\end{aligned} \tag{11.26}$$

In contrast to the J_3 estimate, here the main contribution comes from the first integral on the right-hand side of (11.26), which we split, for $R_0\geq K>0$,

$$\begin{aligned}&\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_{\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-1}^{-\rho(A(\theta)+2\delta)}dt(\dots)\\&\qquad=\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_{\varepsilon}^{R_0\varepsilon}\frac{d\rho}{\rho^2}\int_{-1}^{-\rho(A(\theta)+2\delta)}dt(\dots)\\&\qquad\quad+\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-1}^{-K\frac{\varepsilon}{\rho}}dt(\dots)\\&\qquad\quad+\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-K\frac{\varepsilon}{\rho}}^{-\delta\frac{\varepsilon}{\rho}}dt(\dots)\\&\qquad\quad+\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-\delta\frac{\varepsilon}{\rho}}^{-\rho(A(\theta)+2\delta)}dt(\dots).\end{aligned} \tag{11.27}$$

Regarding the bounds of integration over t, note that $K\frac{\varepsilon}{\rho}\leq 1$ if $K\leq R_0$ and $\rho\geq R_0\varepsilon$, while $\delta\frac{\varepsilon}{\rho}>\rho(A(\theta)+2\delta)$ only holds when $\rho<\left(\frac{\delta}{A(\theta)+2\delta}\right)^{\frac12}\varepsilon^{\frac12}$.

For the first integral in the right-hand side of (11.27), we use (11.14) to estimate

$$\begin{aligned}&\left|\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_{\varepsilon}^{R_0\varepsilon}\frac{d\rho}{\rho^2}\int_{-1}^{-\rho(A(\theta)+2\delta)}dt(\dots)\right|\\&\leq C\int_{S^{n-2}\cap\{A(\theta)>3\delta\}}d\theta\int_{\varepsilon}^{R_0\varepsilon}\frac{d\rho}{\rho^2}\int_{-1}^{-\rho(A(\theta)+2\delta)}\frac{\rho^2}{\varepsilon}b(\theta,t,r)dt\\&\leq\frac{C}{\varepsilon}\int_{\varepsilon}^{R_0\varepsilon}d\rho\leq CR_0.\end{aligned}$$

It follows that

$$\frac{1}{|\ln\varepsilon|}\left|\int_{\mathbb{R}} d\xi\dot{\phi}(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{\varepsilon}^{R_0\varepsilon}\frac{d\rho}{\rho^2}\int_{-1}^{-\rho(A(\theta)+2\delta)} dt(\dots)\right| \le \frac{CR_0}{|\ln\varepsilon|}. \tag{11.28}$$

For the second integral in (11.27), by (11.6) and the monotonicity of ϕ, we have

$$\begin{aligned}
0 &\le \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-1}^{-K\frac{\varepsilon}{\rho}}\\
&\qquad \left\{\phi\left(\xi+\frac{\rho}{\varepsilon}\left(t+\rho b(\theta,t,r)\right)\right)-\phi\left(\xi+\frac{t\rho}{\varepsilon}\right)\right\}\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\\
&\le \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-1}^{-K\frac{\varepsilon}{\rho}}\\
&\qquad \left\{\phi\left(\xi+\frac{t\rho}{\varepsilon}+\frac{\rho^2}{\varepsilon}(A(\theta)+C_0r)\right)-\phi\left(\xi+\frac{t\rho}{\varepsilon}\right)\right\}\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\\
&\le \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-1}^{-K\frac{\varepsilon}{\rho}} dt\int_0^1\\
&\qquad \dot{\phi}\left(\xi+\frac{t\rho}{\varepsilon}+\tau\frac{\rho^2}{\varepsilon}(A(\theta)+C_0r)\right)\frac{\rho^2}{\varepsilon}(A(\theta)+C_0r)d\tau\\
&\le \frac{C}{\varepsilon}\int_{S^{n-2}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}} d\rho\int_0^1 d\tau\int_{-1}^{-K\frac{\varepsilon}{\rho}}\partial_t\left(\phi\left(\xi+\frac{\rho t}{\varepsilon}+\tau\frac{\rho^2}{\varepsilon}(A(\theta)+C_0r)\right)\right)\frac{\varepsilon}{\rho}dt\\
&= C\int_{S^{n-2}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho}\int_0^1\\
&\qquad \left\{\phi\left(\xi-K+\tau\frac{\rho^2}{\varepsilon}(A(\theta)+C_0r)\right)-\phi\left(\xi-\frac{\rho}{\varepsilon}+\tau\frac{\rho^2}{\varepsilon}(A(\theta)+C_0r)\right)\right\}d\tau.
\end{aligned}$$

Now, if

$$R_0\varepsilon<\rho<(R\varepsilon)^{\frac12},\quad |\tau|<1,\quad |\xi|<\frac{K}{2},\quad 4CR\le K\le R_0,$$

then

$$\xi-K+\tau\frac{\rho^2}{\varepsilon}(A(\theta)+C_0r)\le -K+|\xi|+CR\le-\frac{K}{4},$$

and

$$\xi-\frac{\rho}{\varepsilon}+\tau\frac{\rho^2}{\varepsilon}(A(\theta)+C_0r)\le -R_0+|\xi|+CR\le-\frac{K}{4}.$$

Therefore, by (4.2), we have

$$\phi\left(\xi-K+\tau\frac{\rho^2}{\varepsilon}(A(\theta)+C_0r)\right),\ \phi\left(\xi-\frac{\rho}{\varepsilon}+\tau\frac{\rho^2}{\varepsilon}(A(\theta)+C_0r)\right)\le\frac{C}{K},$$

which implies that

$$\begin{aligned}
&\int_{S^{n-2}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho}\int_0^1\left\{\phi\left(\xi-K\frac{\rho^2}{\varepsilon}(A(\theta)+C_0r)\right)-\phi\left(\xi-\frac{\rho}{\varepsilon}+\frac{\rho^2}{\varepsilon}(A(\theta)+C_0r)\right)\right\}d\tau\\
&\le\frac{C}{K}\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho}\ \le\frac{C|\ln\varepsilon|}{K}.
\end{aligned}$$

The computations above yield

$$\left|\int_{\{|\xi|<\frac{K}{2}\}} d\xi\dot\phi(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-1}^{-K\frac{\varepsilon}{\rho}} dt(\dots)\right| \quad\le \frac{C|\ln\varepsilon|}{K}.$$

On the other hand, estimating as above but using that $0<\phi<1$ and (11.13), we obtain

$$\left|\int_{\{|\xi|>\frac{K}{2}\}} d\xi\dot\phi(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-1}^{-K\frac{\varepsilon}{\rho}} dt(\dots)\right|$$
$$\le C\int_{\{|\xi|>\frac{K}{2}\}} d\xi\dot\phi(\xi)\int_{S^{n-2}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho}\int_0^{\tau} d\tau\le\frac{C|\ln\varepsilon|}{K}.$$

We conclude that

$$\frac{1}{|\ln\varepsilon|}\left|\int_{\mathbb R} d\xi\dot\phi(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-1}^{-K\frac{\rho}{\varepsilon}} dt(\dots)\right|\le\frac{C}{K}. \tag{11.29}$$

We next estimate the third term on the right-hand side of (11.27). We first notice that, if $|t|\le K\frac{\varepsilon}{\rho}\le 1$ and $\rho>R_0\varepsilon$, then

$$\frac{1}{(t^2+1)^{\frac{n+1}{2}}}=1+O\left(\frac{K}{R_0}\right).$$

We set

$$R_0=K^2,$$

then

$$\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-K\frac{\varepsilon}{\rho}}^{-\delta\frac{\varepsilon}{\rho}}\left\{\phi\left(\xi+\frac{\rho}{\varepsilon}(t+\rho b(\theta,t,r))\right)-\phi\left(\xi+\frac{t\rho}{\varepsilon}\right)\right\}\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}$$
$$=\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-K\frac{\varepsilon}{\rho}}^{-\delta\frac{\varepsilon}{\rho}}(\dots)\left(1+O(K^{-1})\right)dt.$$

Using again (11.6) and the monotonicity of ϕ, we get

$$\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-K\frac{\varepsilon}{\rho}}^{-\delta\frac{\varepsilon}{\rho}}\left\{\phi\left(\xi+\frac{\rho}{\varepsilon}(t+\rho b(\theta,t,r))\right)-\phi\left(\xi+\frac{t\rho}{\varepsilon}\right)\right\}dt$$
$$\le\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-K\frac{\varepsilon}{\rho}}^{-\delta\frac{\varepsilon}{\rho}}\left\{\phi\left(\xi+\frac{t\rho}{\varepsilon}+\frac{\rho^2}{\varepsilon}(A(\theta)+C_0r)\right)-\phi\left(\xi+\frac{t\rho}{\varepsilon}\right)\right\}dt$$
$$=\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-K\frac{\varepsilon}{\rho}}^{-\delta\frac{\varepsilon}{\rho}} dt\int_0^{\tau}$$
$$\dot\phi\left(\xi+\frac{t\rho}{\varepsilon}+\tau\frac{\rho^2}{\varepsilon}(A(\theta)+C_0r)\right)\frac{\rho^2}{\varepsilon}(A(\theta)+C_0r)d\tau$$
$$=\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho}\int_0^{\tau} d\tau\int_{-K\frac{\varepsilon}{\rho}}^{-\delta\frac{\varepsilon}{\rho}}$$

$$\partial_t \left(\phi \left(\xi + \frac{t\rho}{\varepsilon} + \tau \frac{\rho^2}{\varepsilon}(A(\theta) + C_0 r) \right) \right) (A(\theta) + C_0 r) dt.$$

Similarly,

$$\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{d\rho}{\rho^2} \int_{-K\frac{\varepsilon}{\rho}}^{-\delta\frac{\varepsilon}{\rho}} \left\{ \phi\left(\xi + \frac{\rho}{\varepsilon}\left(t + \rho b(\theta, t, r)\right)\right) - \phi\left(\xi + \frac{t\rho}{\varepsilon}\right) \right\} dt$$
$$\geq \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{d\rho}{\rho} \int_0^\tau d\tau \int_{-K\frac{\varepsilon}{\rho}}^{-\delta\frac{\varepsilon}{\rho}}$$
$$\partial_t \left(\phi \left(\xi + \frac{t\rho}{\varepsilon} + \tau \frac{\rho^2}{\varepsilon}(A(\theta) - C_0 r) \right) \right) (A(\theta) - C_0 r) dt.$$

For the main terms in the integrals above, we use (11.14) to write

$$\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta) d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{d\rho}{\rho} \int_0^\tau d\tau \int_{-K\frac{\varepsilon}{\rho}}^{-\delta\frac{\varepsilon}{\rho}} \partial_t \left(\phi \left(\xi + \frac{t\rho}{\varepsilon} + \tau \frac{\rho^2}{\varepsilon}(A(\theta) \pm C_0 r) \right) \right) dt$$
$$= \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta) d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{d\rho}{\rho} \int_0^\tau$$
$$\left\{ \phi \left(\xi - \delta + \tau \frac{\rho^2}{\varepsilon}(A(\theta) \pm C_0 r) \right) - \phi \left(\xi - K + \tau \frac{\rho^2}{\varepsilon}(A(\theta) \pm C_0 r) \right) \right\} d\tau$$
$$= \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta) d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \left\{ \phi(\xi - \delta) - \phi(\xi - K) + O\left(\frac{\rho^2}{\varepsilon}\right) \right\} \frac{d\rho}{\rho}.$$

As above, we find that

$$\int_{\mathbb{R}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta) d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \phi(\xi - K) \frac{d\rho}{\rho} \leq \frac{C|\ln\varepsilon|}{K}.$$

Moreover, regarding the error term,

$$\int_{\mathbb{R}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta) d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{d\rho}{\rho} \frac{\rho^2}{\varepsilon} \leq CR.$$

The main contribution comes from the following integral

$$\int_{\mathbb{R}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta) d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \phi(\xi - \delta) \frac{d\rho}{\rho}$$
$$= \int_{\mathbb{R}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta) d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \phi(\xi) \frac{d\rho}{\rho} + O(\delta|\ln\varepsilon|)$$
$$= \int_{\mathbb{R}} \frac{1}{2} \frac{d}{d\xi} \left(\phi^2(\xi)\right) d\xi \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta) d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{d\rho}{\rho} + O(\delta|\ln\varepsilon|)$$
$$= \frac{1}{2} \left(\frac{1}{2}|\ln\varepsilon| + \frac{1}{2}\ln(R) - \ln(R_0) \right) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta) d\theta + O(\delta|\ln\varepsilon|),$$

where we used (11.14) and that $\phi(\infty) = 1$ and $\phi(-\infty) = 0$. Putting it all together, we get

$$\begin{aligned} &\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} d\xi\dot{\phi}(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-K\frac{\varepsilon}{\rho}}^{-\delta\frac{\varepsilon}{\rho}} dt(\ldots) \\ &\quad = \frac14\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta)d\theta + o_\varepsilon(1) + O(r) + O(K^{-1}) + O(\delta). \end{aligned} \tag{11.30}$$

Finally, for the fourth term in (11.27), we use (11.14) to estimate

$$\begin{aligned} &\left|\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-\delta\frac{\varepsilon}{\rho}}^{-\rho(A(\theta)+2\delta)} dt(\ldots)\right| \\ &\quad \le C\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\left|\delta\frac{\varepsilon}{\rho} - \rho(A(\theta)+2\delta)\right|\frac{\rho^2}{\varepsilon}\frac{d\rho}{\rho^2} \\ &\quad \le C\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\left(\frac{\delta}{\rho}+\frac{\rho}{\varepsilon}\right)d\rho \\ &\quad \le C(\delta|\ln\varepsilon| + R), \end{aligned}$$

which gives

$$\frac{1}{|\ln\varepsilon|}\left|\int_{\mathbb{R}} d\xi\dot{\phi}(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-\delta\frac{\varepsilon}{\rho}}^{-\rho(A(\theta)+2\delta)} dt(\ldots)\right| \le C\left(\delta + \frac{R}{|\ln\varepsilon|}\right). \tag{11.31}$$

Choosing

$$K = \delta^{-1},$$

from (11.28), (11.29), (11.30) and (11.31) we get

$$\begin{aligned} &\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} d\xi\dot{\phi}(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{-1}^{-\rho(A(\theta)+2\delta)} dt(\ldots) \\ &\quad = \frac14\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta)d\theta + o_\varepsilon(1) + O(r) + O(\delta). \end{aligned} \tag{11.32}$$

To prove (11.24), it remains to estimate the second integral on the right-hand side of (11.26). For that, we estimate,
(11.33)

$$\begin{aligned}
0 &\leq \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac12}}^{r} \frac{d\rho}{\rho^2} \int_{-1}^{-\rho(A(\theta)+2\delta)} dt(\dots)\\
&\leq \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac12}}^{r} \frac{d\rho}{\rho^2} \int_{-1}^{-\rho(A(\theta)+2\delta)}\\
&\qquad \left\{\phi\left(\xi+\frac{t\rho}{\varepsilon}+\frac{\rho^2}{\varepsilon}(A(\theta)+C_0r)\right)-\phi\left(\xi+\frac{t\rho}{\varepsilon}\right)\right\}dt\\
&= \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac12}}^{r} \frac{d\rho}{\rho^2} \int_{-1}^{-\rho(A(\theta)+2\delta)} dt \int_0^1\\
&\qquad \dot\phi\left(\xi+\frac{t\rho}{\varepsilon}+\tau\frac{\rho^2}{\varepsilon}(A(\theta)+C_0r)\right)\frac{\rho^2(A(\theta)+C_0r)}{\varepsilon}d\tau\\
&\leq \frac{C}{\varepsilon}\int_{S^{n-2}} d\theta \int_{(R\varepsilon)^{\frac12}}^{r} d\rho \int_0^1 d\tau \int_{-1}^{-\rho(A(\theta)+2\delta)} \partial_t\left(\phi\left(\xi+\frac{t\rho}{\varepsilon}+\tau\frac{\rho^2}{\varepsilon}(A(\theta)+C_0r)\right)\right)\frac{\varepsilon}{\rho}dt\\
&= C\int_{S^{n-2}} d\theta \int_{(R\varepsilon)^{\frac12}}^{r} \frac{d\rho}{\rho}\int_0^1\left\{\phi\left(\xi+\frac{\rho^2}{\varepsilon}[-A(\theta)-2\delta+\tau(A(\theta)+C_0r)]\right)\right.\\
&\qquad \left.-\phi\left(\xi-\frac{\rho}{\varepsilon}+\tau\frac{\rho^2}{\varepsilon}(A(\theta)+C_0r)\right)\right\}d\tau.
\end{aligned}$$

Now, if

$$\theta\in S^{n-1},\quad (R\varepsilon)^{\frac12}<\rho<r,\quad |\tau|<1,\quad |\xi|<\frac{R\delta}{2},$$

since $C_0r<\delta$, then

$$\xi+\frac{\rho^2}{\varepsilon}[-A(\theta)-2\delta+\tau(A(\theta)+C_0r)]\leq|\xi|-\frac{\delta\rho^2}{\varepsilon}\leq\frac{R\delta}{2}-R\delta\leq-\frac{R\delta}{2},$$

and for r,ε sufficiently small,

$$\xi-\frac{\rho}{\varepsilon}+\tau\frac{\rho^2}{\varepsilon}(A(\theta)+C_0r)\leq|\xi|-\frac{\rho}{2\varepsilon}\leq -R.$$

Choose

$$R=\delta^{-2},$$

then, by (4.2),

$$\phi\left(\xi+\frac{\rho^2}{\varepsilon}[-A(\theta)-2\delta+\tau(A(\theta)+C_0r)]\right),\,\phi\left(\xi-\frac{\rho}{\varepsilon}+\tau\frac{\rho^2}{\varepsilon}(A(\theta)+C_0r)\right)\leq C\delta.$$

The computations above yield

$$\left|\int_{\{|\xi|<\frac{R\delta}{2}\}} d\xi\dot\phi(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac12}}^{r} \frac{d\rho}{\rho^2} \int_{-1}^{-\rho(A(\theta)+2\delta)} dt(\dots)\right|\quad\leq C\delta|\ln\varepsilon|.$$

On the other hand, by (11.13) and (11.33),

$$\left|\int_{\{|\xi|>\frac{R\delta}{2}\}} d\xi\dot\phi(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac12}}^{r} \frac{d\rho}{\rho^2} \int_{-1}^{-\rho(A(\theta)+2\delta)} dt(\dots)\right|$$

$$\leq C \int_{\{|\xi|>\frac{R\delta}{2}\}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}} d\theta \int_{(R\varepsilon)^{\frac{1}{2}}}^{r} \frac{d\rho}{\rho} \int_0^{\tau} d\tau$$
$$\leq C\delta |\ln \varepsilon|.$$

We conclude that

$$\frac{1}{|\ln \varepsilon|} \left| \int_{\mathbb{R}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac{1}{2}}}^{r} \frac{d\rho}{\rho^2} \int_{-1}^{-\rho(A(\theta)+2\delta)} dt(\dots) \right| \leq C\delta. \tag{11.34}$$

From (11.32) and (11.34), estimate (11.24) follows.

We now check that the estimate for J_5 in (11.25) holds. We write

$$\begin{aligned} J_5(\xi) = & \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{d\rho}{\rho^2} \int_{\delta\rho}^{1} dt(\dots) \\ & + \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{(R\varepsilon)^{\frac{1}{2}}}^{r} \frac{d\rho}{\rho^2} \int_{\delta\rho}^{1} dt(\dots), \end{aligned} \tag{11.35}$$

and for $R_0 \geq K > 0$,

(11.36)
$$\begin{aligned} \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{d\rho}{\rho^2} \int_{\delta\rho}^{1} dt(\dots) = & \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{\varepsilon}^{R_0\varepsilon} \frac{d\rho}{\rho^2} \int_{\delta\rho}^{1} dt(\dots) \\ & + \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{d\rho}{\rho^2} \int_{\delta\rho}^{\delta\frac{\varepsilon}{\rho}} dt(\dots) \\ & + \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{d\rho}{\rho^2} \int_{\delta\frac{\varepsilon}{\rho}}^{K\frac{\varepsilon}{\rho}} dt(\dots) \\ & + \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{d\rho}{\rho^2} \int_{K\frac{\varepsilon}{\rho}}^{1} dt(\dots). \end{aligned}$$

Similar computations as for the estimates (11.28), (11.29) and (11.31) yield

$$\frac{1}{|\ln \varepsilon|} \left| \int_{\mathbb{R}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{\varepsilon}^{R_0\varepsilon} \frac{d\rho}{\rho^2} \int_{\delta\rho}^{1} dt(\dots) \right| \leq \frac{CR_0}{|\ln \varepsilon|}, \tag{11.37}$$

$$\frac{1}{|\ln \varepsilon|} \left| \int_{\mathbb{R}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{d\rho}{\rho^2} \int_{K\frac{\rho}{\varepsilon}}^{1} dt(\dots) \right| \leq \frac{C}{K}, \tag{11.38}$$

and

$$\frac{1}{|\ln \varepsilon|} \left| \int_{\mathbb{R}} d\xi \dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{d\rho}{\rho^2} \int_{\delta\rho}^{\delta\frac{\varepsilon}{\rho}} dt(\dots) \right| \leq C\left(\delta + \frac{R\delta}{|\ln \varepsilon|}\right). \tag{11.39}$$

The third term on the right-hand side of (11.36) is similar to (11.30). Indeed, as above we get

(11.40)

$$\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{d\rho}{\rho^2} \int_{\delta\frac{\varepsilon}{\rho}}^{K\frac{\varepsilon}{\rho}} dt(\dots)$$
$$= \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta)d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \{\phi(\xi+K)-\phi(\xi+\delta)\}\frac{d\rho}{\rho} + O(R) + O(r|\ln\varepsilon|).$$

By (4.2), for $|\xi|<\frac{K}{2}$,

$$\phi(\xi+K) = 1 + O\left(\frac{1}{K}\right).$$

Therefore,

$$\int_{\{|\xi|<\frac{K}{2}\}} d\xi\dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{d\rho}{\rho^2} \int_{\delta\frac{\varepsilon}{\rho}}^{K\frac{\varepsilon}{\rho}} dt(\dots)$$
$$= \int_{\{|\xi|<\frac{K}{2}\}} d\xi\dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta)d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \left\{1+O(K^{-1}))-\phi(\xi+\delta)\right\}\frac{d\rho}{\rho}$$
$$+ O(R) + O(r|\ln\varepsilon|)$$
$$= \int_{\{|\xi|<\frac{K}{2}\}} d\xi\dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta)d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \left\{1-\phi(\xi)+O(K^{-1})+O(\delta)\right\}\frac{d\rho}{\rho}$$
$$+ O(R) + O(r|\ln\varepsilon|)$$
$$= \int_{\{|\xi|<\frac{K}{2}\}} \left\{\dot{\phi}(\xi) - \frac{1}{2}\frac{d}{d\xi}(\phi(\xi))^2\right\} d\xi \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta)d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{d\rho}{\rho}$$
$$+ O(K^{-1}|\ln\varepsilon|) + O(\delta|\ln\varepsilon|) + O(R) + O(r|\ln\varepsilon|)$$
$$= \left[\phi\left(\frac{K}{2}\right) - \phi\left(-\frac{K}{2}\right) - \frac{1}{2}\left(\phi^2\left(\frac{K}{2}\right) - \phi^2\left(-\frac{K}{2}\right)\right)\right] \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta)d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{d\rho}{\rho}$$
$$+ O(K^{-1}|\ln\varepsilon|) + O(\delta|\ln\varepsilon|) + O(R) + O(r|\ln\varepsilon|).$$

Since, again by (4.2),

$$\phi\left(\frac{K}{2}\right) - \phi\left(-\frac{K}{2}\right) - \frac{1}{2}\left(\phi^2\left(\frac{K}{2}\right) - \phi^2\left(-\frac{K}{2}\right)\right) = \frac{1}{2} + O(K^{-1}),$$

we get

$$\int_{\{|\xi|<\frac{K}{2}\}} d\xi\dot{\phi}(\xi) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{d\rho}{\rho^2} \int_{\delta\frac{\varepsilon}{\rho}}^{K\frac{\varepsilon}{\rho}} dt(\dots)$$
$$= \left(\frac{1}{2}+O(K^{-1})\right) \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta)d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac{1}{2}}} \frac{d\rho}{\rho}$$
$$+ O(K^{-1}|\ln\varepsilon|) + O(\delta|\ln\varepsilon|) + O(R) + O(r|\ln\varepsilon|)$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta)d\theta \int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}} \frac{d\rho}{\rho} + O(K^{-1}|\ln\varepsilon|) + O(\delta|\ln\varepsilon|) + O(R) + O(r)\\
&= \frac{1}{2}\left(\frac12|\ln\varepsilon| + \frac12\ln(R) - \ln(R_0)\right)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta)d\theta\\
&\quad + O(K^{-1}|\ln\varepsilon|) + O(\delta|\ln\varepsilon|) + O(R) + O(r|\ln\varepsilon|).
\end{aligned}
$$

On the other hand, by (11.13) and (11.40),

$$
\begin{aligned}
&\int_{\{|\xi|>\frac{K}{2}\}} d\xi\dot\phi(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{\delta\frac{\varepsilon}{\rho}}^{K\frac{\varepsilon}{\rho}} dt(\dots)\\
&\quad= \int_{\{|\xi|>\frac{K}{2}\}} d\xi\dot\phi(\xi)\left[\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta)d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\{\phi(\xi+K)-\phi(\xi+\delta)\}\frac{d\rho}{\rho}\right.\\
&\qquad\left. + O(R) + O(r|\ln\varepsilon|)\right]\\
&\quad= O(K^{-1}|\ln\varepsilon|) + O(K^{-1}R).
\end{aligned}
$$

The two estimates above give

$$
\begin{aligned}
\frac{1}{|\ln\varepsilon|}&\int_{\mathbb{R}} d\xi\dot\phi(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{R_0\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{\delta\frac{\varepsilon}{\rho}}^{K\frac{\varepsilon}{\rho}} dt(\dots)\\
&= \frac14\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta)d\theta + o_\varepsilon(1) + O(r) + O(K^{-1}) + O(\delta).
\end{aligned}
\tag{11.41}
$$

Choosing $K=\delta^{-1}$, from (11.36), (11.37), (11.38), (11.41) and (11.39) we get

$$
\begin{aligned}
\frac{1}{|\ln\varepsilon|}&\int_{\mathbb{R}} d\xi\dot\phi(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{\varepsilon}^{(R\varepsilon)^{\frac12}}\frac{d\rho}{\rho^2}\int_{\delta\rho}^{1} dt(\dots)\\
&= \frac14\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta)d\theta + o_\varepsilon(1) + O(r) + O(\delta).
\end{aligned}
\tag{11.42}
$$

Finally, for the second term in the right-hand side of (11.35), we estimate as in (11.34) and choose $R=\delta^{-2}$ to obtain

$$
\frac{1}{|\ln\varepsilon|}\left|\int_{\mathbb{R}} d\xi\dot\phi(\xi)\int_{S^{n-2}\cap\{A(\theta)>3\delta\}} d\theta\int_{(R\varepsilon)^{\frac12}}^{r}\frac{d\rho}{\rho^2}\int_{\delta\rho}^{1} dt(\dots)\right| \le C\delta.
\tag{11.43}
$$

From (11.35), (11.42) and (11.43), estimate (11.25) follows. Collecting the estimates for $\int_{\mathbb{R}}\dot\phi(\xi)J_i(\xi)\,d\xi$, $1\le i\le 5$ in (11.15), (11.22), (11.23), (11.24), and (11.25), we can write the corresponding expression for $\int_{\mathbb{R}}\dot\phi(\xi)I_1^1(\xi)\,d\xi$ with $I_1^1(\xi)$ in (11.11) as

$$
\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot\phi(\xi)I_1^1(\xi)\,d\xi = \int_{S^{n-2}\cap\{A(\theta)>3\delta\}} A(\theta)d\theta + o_\varepsilon(1) + O(\delta) + O(r).
\tag{11.44}
$$

In the same way, we obtain

$$
\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot\phi(\xi)I_1^2(\xi)\,d\xi = \int_{S^{n-2}\cap\{A(\theta)<-3\delta\}} A(\theta)d\theta + o_\varepsilon(1) + O(\delta) + O(r).
\tag{11.45}
$$

Finally, let us show

$$
\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot\phi(\xi)I_1^3(\xi)\,d\xi = o_\varepsilon(1) + o_\delta(1) + O(r).
\tag{11.46}
$$

If one of the eigenvalues λ_i is different than zero, then $\mathcal{H}^{n-2}(\{\theta \in S^{n-2} \,|\, A(\theta) = 0\}) = 0$, where $\mathcal{H}^{n-2}$ is the $n-2$ dimensional Hausdorff measure. In particular, $\mathcal{H}^{n-2}(\{\theta \in S^{n-2} \,|\, |A(\theta)| < 3\delta\}) = o_\delta(1)$. Therefore, integrating in t as before,

$$\begin{aligned}\left|\int_{\mathbb{R}} \dot\phi(\xi) I_1^3(\xi)\, d\xi\right| &\le C\int_{\mathbb{R}} d\xi \dot\phi(\xi) \int_{S^{n-2}\cap\{|A(\theta)|<3\delta\}} d\theta \int_\varepsilon^r \frac{d\rho}{\rho} \\ &\le C|\ln\varepsilon| \int_{S^{n-2}\cap\{|A(\theta)|<3\delta\}} d\theta = |\ln\varepsilon| o_\delta(1),\end{aligned}$$

which implies (11.46).

If instead, $\lambda_i = 0$ for all $i = 1, \ldots, n-1$, then $A(\theta) \equiv 0$ and $S^{n-2} \cap \{|A(\theta)| < 3\delta\} = S^{n-2}$. In this case, we write

$$\begin{aligned} I_1^3(\xi) = &\int_{S^{n-2}} d\theta \int_\varepsilon^r \frac{d\rho}{\rho^2} \int_{-1}^{-2\delta\rho} dt(\ldots) + \int_{S^{n-2}} d\theta \int_\varepsilon^r \frac{d\rho}{\rho^2} \int_{-2\delta\rho}^{2\delta\rho} dt(\ldots) \\ &+ \int_{S^{n-2}} d\theta \int_\varepsilon^r \frac{d\rho}{\rho^2} \int_{2\delta\rho}^{1} dt(\ldots). \end{aligned}$$

As for the estimates of $\int_{\mathbb{R}} \dot\phi(\xi) J_1(\xi) d\xi$ and $\int_{\mathbb{R}} \dot\phi(\xi) J_5(\xi) d\xi$, (11.24) and (11.25), we get

$$\begin{aligned}\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} d\xi\dot\phi(\xi) \int_{S^{n-2}} d\theta \int_\varepsilon^r \frac{d\rho}{\rho^2} \int_{-1}^{-2\delta\rho} dt(\ldots) &= \frac{1}{4}\int_{S^{n-2}} A(\theta) d\theta + o_\varepsilon(1) + O(r) + O(\delta) \\ &= o_\varepsilon(1) + O(r) + O(\delta),\end{aligned}$$

and

$$\begin{aligned}\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} d\xi\dot\phi(\xi) \int_{S^{n-2}} d\theta \int_\varepsilon^r \frac{d\rho}{\rho^2} \int_{2\delta\rho}^{1} dt(\ldots) &= \frac{1}{4}\int_{S^{n-2}} A(\theta) d\theta + o_\varepsilon(1) + O(r) + O(\delta) \\ &= o_\varepsilon(1) + O(r) + O(\delta).\end{aligned}$$

Moreover, similarly to the estimates of $\int_{\mathbb{R}} \dot\phi(\xi) J_2(\xi) d\xi$ and $\int_{\mathbb{R}} \dot\phi(\xi) J_4(\xi) d\xi$, (11.22) and (11.23),

$$\frac{1}{|\ln\varepsilon|}\left|\int_{\mathbb{R}} d\xi\dot\phi(\xi) \int_{S^{n-2}} d\theta \int_\varepsilon^r \frac{d\rho}{\rho^2} \int_{-2\delta\rho}^{2\delta\rho} dt(\ldots)\right| \le C\delta.$$

Estimate (11.46) then follows.

From (11.44), (11.45) and (11.46) we finally obtain (11.10).

Step 1b. Estimating $\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} \dot\phi(\xi) I_2(\xi)\, d\xi$. We will show that

$$\frac{1}{|\ln\varepsilon|}\left|\int_{\mathbb{R}} \dot\phi(\xi) I_2(\xi) d\xi\right| \le o_r(1). \tag{11.47}$$

Recalling (11.5), we see that for $|t| > 1$, there is $C_1 > 0$ such that

$$A(\theta) - C_1 r t^2 \le b(\theta, t, r) \le A(\theta) + C_1 r t^2. \tag{11.48}$$

Then, for C_1 as above and $R > 2$, to be determined, we write

$$\begin{aligned} I_2(\xi) = &\int_\varepsilon^r \frac{d\rho}{\rho^2} \int_{S^{n-2}} d\theta \int_{\left\{\frac{1}{2\sqrt{rRC_1\rho}} < |t| < \frac{r}{\rho}\right\}} dt(\ldots) + \int_\varepsilon^r \frac{d\rho}{\rho^2} \int_{S^{n-2}} d\theta \int_{\left\{1 < |t| < \frac{1}{2\sqrt{rRC_1\rho}}\right\}} dt(\ldots) \\ =&: I_2^1(\xi) + I_2^2(\xi). \end{aligned}$$

We first estimate

$$
\begin{aligned}
|I_2^1(\xi)| &\le 2\int_\varepsilon^r \frac{d\rho}{\rho^2}\int_{S^{n-2}} d\theta \int_{\left\{|t|>\frac{1}{2\sqrt{rRC_1\rho}}\right\}} \frac{dt}{|t|^{n+1}}\\
&\le C\int_\varepsilon^r (rR)^{\frac{n}{2}}\rho^{\frac{n}{2}-2}d\rho \le C(rR)^{\frac{n}{2}}\int_\varepsilon^r \frac{d\rho}{\rho} \le C(rR)^{\frac{n}{2}}|\ln\varepsilon|,
\end{aligned}
$$

so that

$$
\frac{1}{|\ln\varepsilon|}\left|\int_{\mathbb{R}} \dot\phi(\xi) I_2^1(\xi)d\xi\right| \le C(rR)^{\frac{n}{2}}. \tag{11.49}
$$

Next, let us estimate $\int_{\mathbb{R}}\dot\phi(\xi)I_2^2(\xi)d\xi$. If

$$
|\xi| < \frac{|t|\rho}{\varepsilon} - \frac{\rho}{R\varepsilon}, \quad 0<\rho<r, \quad 1<|t|<\frac{1}{2\sqrt{rRC_1\rho}}, \quad 0<r|A(\theta)|<\frac{1}{4R}, \quad R>2,
$$

then

$$
\begin{aligned}
\left|\xi + \frac{t\rho}{\varepsilon} + \frac{\rho^2}{\varepsilon} b(\theta,t,r)\right| &\ge \frac{|t|\rho}{\varepsilon} - |\xi| - \frac{\rho^2}{\varepsilon}|A(\theta)| - \frac{\rho^2}{\varepsilon} C_1 r t^2\\
&\ge \frac{|t|\rho}{\varepsilon} - |\xi| - \frac{\rho}{4R\varepsilon} - \frac{\rho}{4R\varepsilon} \ge \frac{\rho}{2R\varepsilon}.
\end{aligned}
$$

Therefore, by (4.3), for some $\tau\in(0,1)$,

$$
\begin{aligned}
\left|\phi\left(\xi + \frac{\rho}{\varepsilon}\left(t+\rho b(\theta,t,r)\right)\right) - \phi\left(\xi+\frac{t\rho}{\varepsilon}\right)\right| &= \phi'\left(\xi + \frac{t\rho}{\varepsilon} + \tau\rho b(\theta,t,r)\right)\frac{\rho^2}{\varepsilon} b(\theta,t,r)\\
&\le C\left(\frac{R\varepsilon}{\rho}\right)^2 \frac{\rho^2}{\varepsilon}(1+rt^2) = C(1+rt^2)R^2\varepsilon
\end{aligned}
$$

from which we find that

$$
\begin{aligned}
&\int_\varepsilon^r \frac{d\rho}{\rho^2}\int_{S^{n-2}} d\theta \int_{\left\{1<|t|<\frac{1}{2\sqrt{rRC_1\varepsilon}}\right\}} \int_{\left\{|\xi|<\frac{|t|\rho}{\varepsilon}-\frac{\rho}{R\varepsilon}\right\}} \dot\phi(\xi)\\
&\qquad \left\{\phi\left(\xi+\frac{\rho}{\varepsilon}\left(t+\rho b(\theta,t,r)\right)\right) - \phi\left(\xi+\frac{t\rho}{\varepsilon}\right)\right\}\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\\
&\le CR^2\varepsilon\int_{\mathbb{R}}\dot\phi(\xi)d\xi\int_{S^{n-2}} d\theta\int_\varepsilon^r \frac{d\rho}{\rho^2}\int_{\left\{1<|t|<\frac{1}{2\sqrt{rRC_1\rho}}\right\}} \frac{1+rt^2}{|t|^{n+1}}dt\\
&\le CR^2\varepsilon\int_\varepsilon^r \frac{d\rho}{\rho^2}\int_{\left\{1<|t|<\frac{1}{2\sqrt{rRC_1\varepsilon}}\right\}} \left(\frac{1}{|t|^{n+1}} + \frac{r}{|t|}\right)dt\\
&\le CR^2\varepsilon\left(1+r|\ln\varepsilon|\right)\int_\varepsilon^r \frac{d\rho}{\rho^2}\\
&\le CR^2\left(1+r|\ln\varepsilon|\right).
\end{aligned}
$$

Consequently,

$$\begin{aligned}(11.50)\quad &\frac{1}{|\ln\varepsilon|}\left|\int_\varepsilon^r \frac{d\rho}{\rho^2}\int_{S^{n-2}}d\theta\int_{\left\{1<|t|<\frac{1}{2\sqrt{rRC_1\varepsilon}}\right\}}\int_{\{|\xi|<\frac{\rho}{\varepsilon}-\frac{\rho}{R\varepsilon}\}}\dot{\phi}(\xi)\right.\\ &\left.\left\{\phi\left(\xi+\frac{\rho}{\varepsilon}\left(t+\rho b(\theta,t,r)\right)\right)\quad-\phi\left(\xi+\frac{t\rho}{\varepsilon}\right)\right\}d\xi\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\right|\\ &\leq CR^2\left(\frac{1}{|\ln\varepsilon|}+r\right).\end{aligned}$$

Next, again by (4.3), for $\tau_1,\,\tau_2\in(-1,1)$,

$$\begin{aligned}\int_{\left\{\frac{|t|\rho}{\varepsilon}-\frac{\rho}{R\varepsilon}<|\xi|<\frac{|t|\rho}{\varepsilon}+\frac{\rho}{R\varepsilon}\right\}}\dot{\phi}(\xi)d\xi &= \left(\phi\left(\frac{|t|\rho}{\varepsilon}+\frac{\rho}{R\varepsilon}\right)-\phi\left(\frac{|t|\rho}{\varepsilon}-\frac{\rho}{R\varepsilon}\right)\right)\\ &\quad+\left(\phi\left(-\frac{|t|\rho}{\varepsilon}+\frac{\rho}{R\varepsilon}\right)-\phi\left(-\frac{|t|\rho}{\varepsilon}-\frac{\rho}{R\varepsilon}\right)\right)\\ &=\phi'\left(\frac{|t|\rho}{\varepsilon}+\tau_1\frac{\rho}{R\varepsilon}\right)\frac{2\rho}{R\varepsilon}+\phi'\left(-\frac{|t|\rho}{\varepsilon}+\tau_2\frac{\rho}{R\varepsilon}\right)\frac{2\rho}{R\varepsilon}\\ &\leq C\left(\frac{\varepsilon}{t\rho}\right)^2\frac{\rho}{R\varepsilon}=C\frac{\varepsilon}{R\rho t^2}.\end{aligned}$$

Therefore, recalling (11.48),

$$\begin{aligned}&\left|\int_\varepsilon^r \frac{d\rho}{\rho^2}\int_{S^{n-2}}d\theta\int_{\left\{1<|t|<\frac{1}{2\sqrt{rRC_1\varepsilon}}\right\}}\int_{\left\{\frac{|t|\rho}{\varepsilon}-\frac{\rho}{R\varepsilon}<|\xi|<\frac{|t|\rho}{\varepsilon}+\frac{\rho}{R\varepsilon}\right\}}\dot{\phi}(\xi)\right.\\ &\qquad\left.\left\{\phi\left(\xi+\frac{\rho}{\varepsilon}\left(t+\rho b(\theta,t,r)\right)\right)-\phi\left(\xi+\frac{t\rho}{\varepsilon}\right)\right\}d\xi\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\right|\\ &\leq C\int_{S^{n-2}}d\theta\int_{|t|>1}dt\frac{|b(t,\theta,r)|}{|t|^{n+1}}\int_\varepsilon^r\frac{\rho^2}{\varepsilon}\frac{d\rho}{\rho^2}\int_{\left\{\frac{|t|\rho}{\varepsilon}-\frac{\rho}{R\varepsilon}<|\xi|<\frac{|t|\rho}{\varepsilon}+\frac{\rho}{R\varepsilon}\right\}}\dot{\phi}(\xi)d\xi\\ &\leq C\int_\varepsilon^r\frac{d\rho}{R\rho}\int_{|t|>1}\frac{dt}{|t|^{n+1}}\\ &\leq C\int_\varepsilon^r\frac{d\rho}{R\rho}\\ &\leq\frac{C|\ln\varepsilon|}{R}.\end{aligned}$$

Consequently,

$$\begin{aligned}(11.51)\quad &\frac{1}{|\ln\varepsilon|}\left|\int_\varepsilon^r \frac{d\rho}{\rho^2}\int_{S^{n-2}}d\theta\int_{\left\{1<|t|<\frac{1}{2\sqrt{rRC_1\varepsilon}}\right\}}\int_{\left\{\frac{|t|\rho}{\varepsilon}-\frac{\rho}{R\varepsilon}<|\xi|<\frac{|t|\rho}{\varepsilon}+\frac{\rho}{R\varepsilon}\right\}}\dot{\phi}(\xi)\right.\\ &\left.\left\{\phi\left(\xi+\frac{\rho}{\varepsilon}\left(t+\rho b(\theta,t,r)\right)\right)-\phi\left(\xi+\frac{t\rho}{\varepsilon}\right)\right\}d\xi\frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\right|\leq\frac{C}{R}.\end{aligned}$$

Finally, if

$$|\xi| > \frac{|t|\rho}{\varepsilon} + \frac{\rho}{R\varepsilon}, \quad 0<\rho<r, \quad 1<|t|<\frac{1}{2\sqrt{rRC_1\rho}}, \quad 0<rA(\theta)<\frac{1}{4R}, \quad R>2,$$

then

$$\left|\xi + \frac{t\rho}{\varepsilon} + \frac{\rho^2}{\varepsilon} b(\theta,t,r)\right| \geq |\xi| - \frac{|t|\rho}{\varepsilon} - \frac{\rho^2}{\varepsilon}|A(\theta)| - \frac{\rho^2}{\varepsilon} C_1 r t^2 \geq \frac{\rho}{2R\varepsilon},$$

and as before, by (4.3),

$$\left|\phi\left(\xi + \frac{\rho}{\varepsilon}\left(t+\rho b(\theta,t,r)\right)\right) - \phi\left(\xi + \frac{t\rho}{\varepsilon}\right)\right| \leq C(1+rt^2)R^2\varepsilon.$$

Therefore,

$$\begin{aligned}
&\int_\varepsilon^r \frac{d\rho}{\rho^2}\int_{S^{n-2}} d\theta \int_{\left\{1<|t|<\frac{1}{2\sqrt{rRC_1\varepsilon}}\right\}} \int_{\{|\xi|>\frac{|t|\rho}{\varepsilon}+\frac{\rho}{R\varepsilon}\}} \dot\phi(\xi) \\
&\qquad \left\{\phi\left(\xi + \frac{\rho}{\varepsilon}\left(t+\rho b(\theta,t,r)\right)\right) - \phi\left(\xi + \frac{t\rho}{\varepsilon}\right)\right\} \frac{dt}{(t^2+1)^{\frac{n+1}{2}}} \\
&\leq CR^2\varepsilon \int_\varepsilon^r \frac{d\rho}{\rho^2} \int_{\left\{1<|t|<\frac{1}{2\sqrt{rRC_1\rho}}\right\}} d\xi \frac{1+rt^2}{|t|^{n+1}} dt \\
&\leq CR^2\left(1+r|\ln\varepsilon|\right),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{|\ln\varepsilon|}\Bigg|\int_\varepsilon^r \frac{d\rho}{\rho^2}\int_{S^{n-2}} d\theta \int_{\left\{1<|t|<\frac{1}{2\sqrt{rRC_1\varepsilon}}\right\}} \int_{\{|\xi|>\frac{|t|\rho}{\varepsilon}+\frac{\rho}{R\varepsilon}\}} \dot\phi(\xi) \\
&\quad \left\{\phi\left(\xi + \frac{\rho}{\varepsilon}\left(t+\rho b(\theta,t,r)\right)\right) - \phi\left(\xi + \frac{t\rho}{\varepsilon}\right)\right\} \frac{dt}{(t^2+1)^{\frac{n+1}{2}}}\Bigg| \leq CR^2\left(\frac{1}{|\ln\varepsilon|}+r\right).
\end{aligned} \tag{11.52}$$

From (11.49), (11.50), (11.51) and (11.52), choosing $R = r^{-\frac{1}{3}}$, (11.47) follows.

Recalling (11.9), we combine (11.10) and (11.47) to conclude Step 1 with

$$\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}} \dot\phi(\xi) I(\xi)\, d\xi = \int_{S^{n-2}} A(\theta)\, d\theta + o_\varepsilon(1) + o_\delta(1) + o_r(1), \tag{11.53}$$

where $o_\varepsilon(1)$ depends on the parameters δ and r.

Step 2. Estimating $\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}}\dot\phi(\xi) II(\xi)\, d\xi$. We will show

$$\frac{1}{|\ln\varepsilon|}\left|\int_{\mathbb{R}} \dot\phi(\xi) II(\xi)\, d\xi\right| \leq \frac{C}{|\ln\varepsilon|}. \tag{11.54}$$

Making the same change of variables as in $I(\xi)$ in Step 1 above, we can write

$$\begin{aligned}
II(\xi) &= \int_{|y'|<\varepsilon}\int_{\varepsilon<|y_n|<r} \left(\phi\left(\xi + \frac{1}{\varepsilon}\left(y_n + A(y') + O(r|y|^2)\right)\right) - \phi\left(\xi + \frac{y_n}{\varepsilon}\right)\right) \frac{dy}{|y|^{n+1}} \\
&= \int_{|y'|<\varepsilon} \frac{dy'}{|y'|^n} \int_{\frac{\varepsilon}{|y'|}<|t|<\frac{r}{|y'|}} \frac{dt}{(t^2+1)^{\frac{n+1}{2}}} \\
&\qquad \left\{\phi\left(\xi + \frac{|y'|}{\varepsilon}\left(t + |y'|A\left(\frac{y'}{|y'|}\right) + |y'|O(r(1+t^2))\right)\right) - \phi\left(\xi + \frac{t|y'|}{\varepsilon}\right)\right\}
\end{aligned}$$

$$= \int_{S^{n-2}} d\theta \int_0^\varepsilon \frac{d\rho}{\rho^2} \int_{\frac{\varepsilon}{\rho}<|t|<\frac{r}{\rho}} \frac{dt}{(t^2+1)^{\frac{n+1}{2}}}$$
$$\left\{ \phi\left(\xi + \frac{\rho}{\varepsilon}\left(t + \rho A(\theta) + \rho O(r(1+t^2))\right)\right) - \phi\left(\xi + \frac{t\rho}{\varepsilon}\right)\right\}.$$

Using the regularity of ϕ and that $A(\theta)$ is bounded, we estimate

$$\begin{aligned}
|II(\xi)| &\le C \int_{S^{n-2}} d\theta \int_0^\varepsilon \frac{d\rho}{\rho^2} \int_{\frac{\varepsilon}{\rho}<|t|<\frac{r}{\rho}} \frac{\rho^2}{\varepsilon} |A(\theta) + O(r(1+t^2))| \frac{dt}{(t^2+1)^{\frac{n+1}{2}}} \\
&= \frac{C}{\varepsilon} \int_{S^{n-2}} d\theta \int_0^\varepsilon d\rho \int_{\frac{\varepsilon}{\rho}<|t|<\frac{r}{\rho}} |A(\theta) + O(r(1+t^2))| \frac{dt}{(t^2+1)^{\frac{n+1}{2}}} \\
&\le \frac{C}{\varepsilon} \int_0^\varepsilon d\rho \int_{1<|t|<\infty} (1 + r(1+t^2)) \frac{dt}{(t^2+1)^{\frac{n+1}{2}}} \le C.
\end{aligned}$$

With this, we have (11.54).

Step 3. Conclusion. From (11.3), (11.53) and (11.54), we first send $\varepsilon \to 0$ and then $r, \delta \to 0$ to arrive at

$$\lim_{\varepsilon\to 0} \bar{a}_\varepsilon(x) = \int_{S^{n-2}} A(\theta)\, d\theta, \tag{11.55}$$

which gives the desired result. Indeed,

$$\int_{S^{n-2}} A(\theta)\, d\theta = \frac{1}{2} \sum_{i=1}^{n-1} \lambda_i \int_{S^{n-2}} \theta_i^2\, d\theta = \frac{1}{2} \sum_{i=1}^{n-1} \lambda_i \frac{|S^{n-2}|}{n-1} = \frac{1}{2} \frac{|S^{n-2}|}{n-1} \operatorname{tr}\left(D^2 d(x)\right).$$

□

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